THE GRAVITY FIELD OF THE EARTH

Torge (1980, p.13) describes the significance of the external gravity field of the Earth by the following points:

1. The external gravity field is the reference system for many measured quantities in geodesy. (eg. theodolite and spirit levelling measurements are made with respect to equipotential surfaces of the Earth's gravity field). The gravity field must be known in order to reduce measured quantities to mathematical reference surfaces such as the ellipsoid.

2. If the distribution of gravity values on the surface of the Earth is known, then in combination with other geodetic measurements, the shape of the surface may be determined.

3. The natural reference surface for heights the geoid is an equipotential surface of the Earth's gravity field which best fits, in the least squares sense, mean sea level.

4. The analysis of the Earth's gravity field yields information on the structure of the interior of the Earth, geodesy thus becoming an auxiliary science of geophysics.

THE GRAVITY FIELD OF THE EARTH

GRAVITATION, GRAVITATIONAL POTENTIAL and GRAVITY.

A body at rest on the Earth's surface experiences the GRAVITATIONAL FORCES exerted by the Earth, Moon, Sun and the planets as well as the CENTRIFUGAL force due to the Earth's rotation. The resultant force on the body is the FORCE OF GRAVITY, a function of position. The force of gravity is often simply called GRAVITY and has both magnitude and direction. On the equator, the gravitational and centrifugal forces act in opposite directions, whilst at the poles, only the gravitational force has an effect. This means that gravity is greater at the poles than at the equator, and since the Earth is a deformable fluid body, it is slightly flattened at the poles.

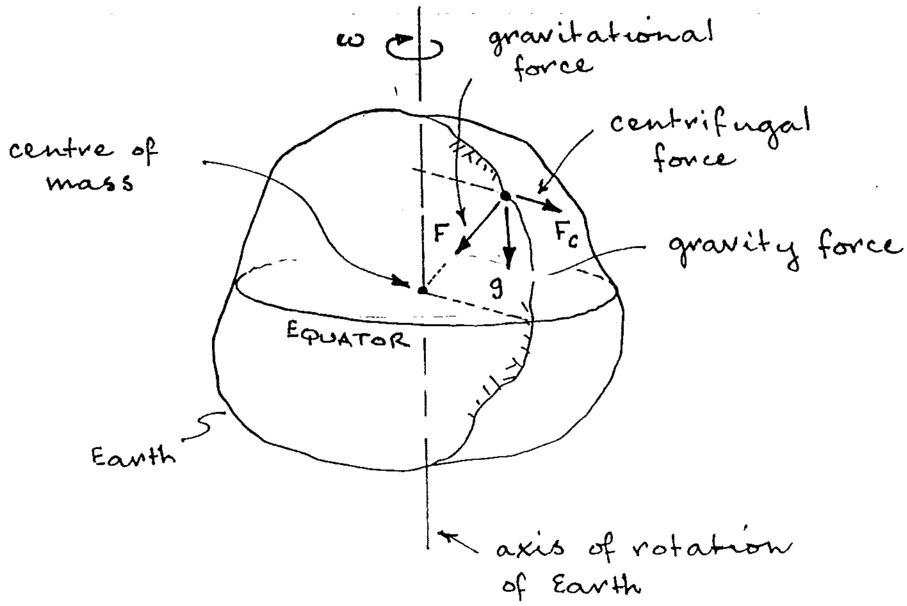


Fig. 1 Gravitational and centrifugal forces

THE GRAVITY FIELD OF THE EARTH

The Earth's gravity field is a VECTOR FIELD meaning that there is a triplet of numbers associated with every point. These numbers representing the XYZ cartesian components of the gravity vector at the point.

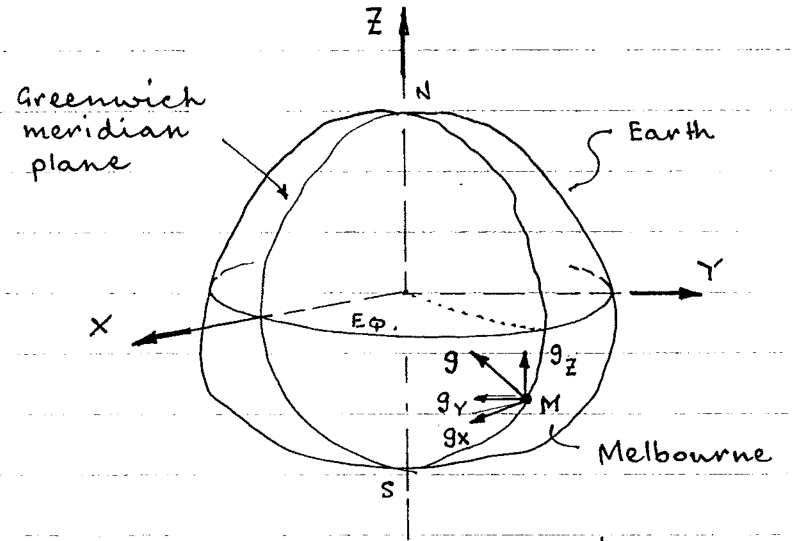


Fig. 2 Gravity vector and components

For example, the value of g and the components for Melbourne (ϕ = S.37°48', λ = E.144°58', h = 0.000) computed using the Geopotential model EGM96 to degree and order 360 are: (ϕ, λ, h are GRS80 values)

$$g = 9.799\,817\,004 \ m/s^2$$

$$g_X = 6.340\,129\,370$$

$$g_Y = -4.445\,087\,147$$

$$g_Z = 6.006\,694\,028 \ m/s^2$$

It is more convenient to work with a SCALAR FIELD where a single-valued function known as the GRAVITY POTENTIAL is assigned to every point and the derivatives of the scalar function are the components of the gravity vector at that point.

THE GRAVITY FIELD OF THE EARTH

In the same way as gravity is the vector sum of the Earth's gravitational and centrifugal forces the the GRAVITY POTENTIAL is the scalar sum (algebraic) of the Earth's GRAVITATIONAL POTENTIAL and CENTRIFUGAL POTENTIAL (or rotational potential)

THE PRINCIPLE OF GRAVITATION

According to Newton's famous Law of Gravitation

> Every particle in the universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

$$F \propto \frac{m_1 m_2}{r^2}$$

or

$$F = G \frac{m_1 m_2}{r^2} \qquad \text{---- (1)}$$

where G is the UNIVERSAL CONSTANT OF GRAVITATION

The most recent value of G is

$$G = 6.673 \times 10^{-11} \; m^3 \, kg^{-1} \, s^{-2}$$

m = metre

kg = kilogram

S = second.

THE GRAVITY FIELD OF THE EARTH

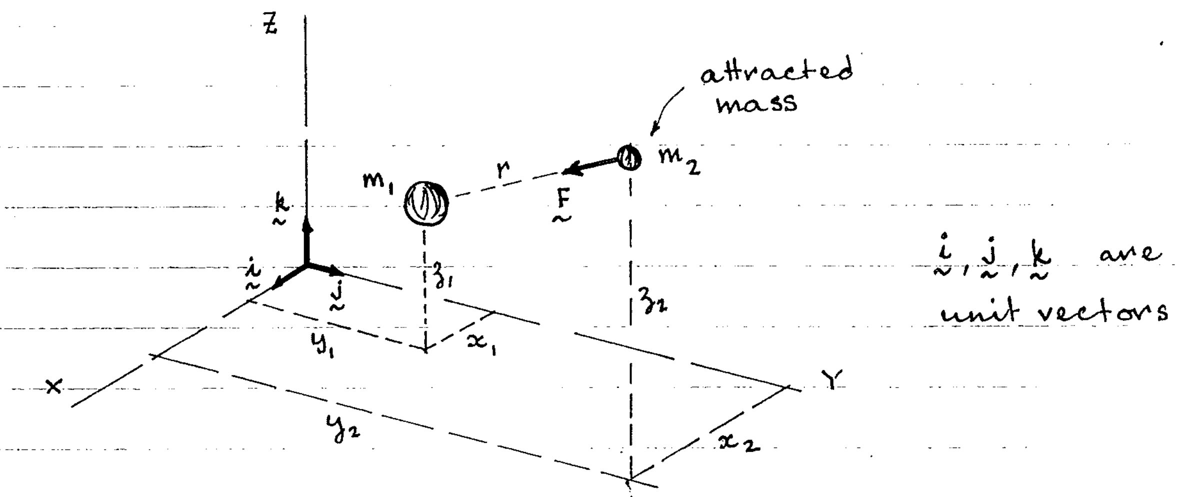


Fig 3 Gravitational force between two particles

The vector equation for the gravitational force $\underset{\sim}{F}$ directed along the line joining m_1 and m_2 is

$$\underset{\sim}{F} = F_x \underset{\sim}{i} + F_y \underset{\sim}{j} + F_z \underset{\sim}{k} \qquad \cdots (2)$$

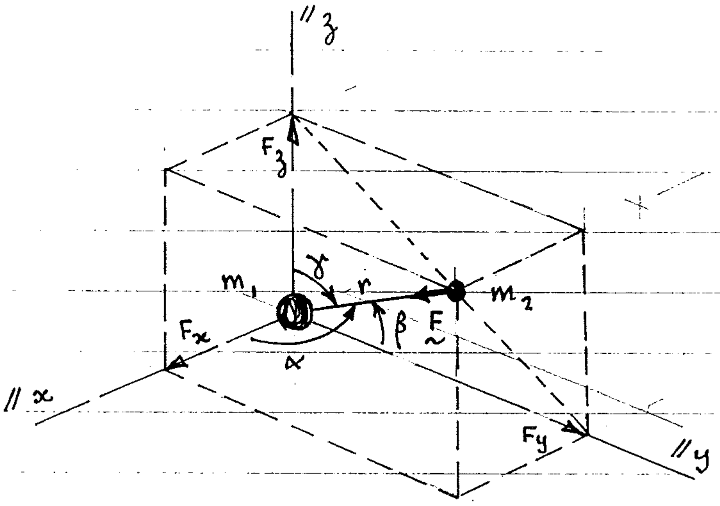
where F_x, F_y, F_z are the cartesian components of $\underset{\sim}{F}$ and the vector $\underset{\sim}{r}$

$$\underset{\sim}{r} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j} + (z_2 - z_1)\underset{\sim}{k} \qquad \cdots (3)$$

has a magnitude

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \cdots (4)$$

THE GRAVITY FIELD OF THE EARTH



$$\cos\alpha = \frac{x_2 - x_1}{r}$$

$$\cos\beta = \frac{y_2 - y_1}{r}$$

$$\cos\gamma = \frac{z_2 - z_1}{r} \qquad \ldots (5)$$

Fig 4 components of gravitational force

The rectangular components of F are

$$F_x = -F\cos\alpha = -G\frac{m_1 m_2}{r^2}\cos\alpha = -G\frac{m_1 m_2}{r^2}\left(\frac{x_2 - x_1}{r}\right)$$

$$F_y = -F\cos\beta = -G\frac{m_1 m_2}{r^2}\cos\beta = -G\frac{m_1 m_2}{r^2}\left(\frac{y_2 - y_1}{r}\right)$$

$$F_z = -F\cos\gamma = -G\frac{m_1 m_2}{r^2}\cos\gamma = -G\frac{m_1 m_2}{r^2}\left(\frac{z_2 - z_1}{r}\right) \qquad \ldots (6)$$

Equations (6) can be simplified by setting the attracted mass m_2 equal to unity and the attracting mass m_1 equal to m. Hence

$$F_x = -\frac{Gm}{r^3}(x_2 - x_1)$$

$$F_y = -\frac{Gm}{r^3}(y_2 - y_1)$$

$$F_z = -\frac{Gm}{r^3}(z_2 - z_1) \qquad \ldots (7)$$

THE GRAVITY FIELD OF THE EARTH.

As an example of equ's (7) consider a point of unit mass at Melbourne (ϕ S. 37°48', λ E. 144°58' h = 0.000 m) on the surface of the GRS 80 ellipsoid (a = 6378137.000 m, f = 1/298.257 222 100 881) whose GM value = $3.986\,005 \times 10^{14}$ m^3/s^2.

X = − 4 131 810.563 m.
Y = 2 896 708.708 m.
Z = − 3 887 927.165 m.
r = 6 370 145.800 m.

Note: GM is the product of the universal constant of gravitation and the mass of the Earth.

$$F_x = -\frac{GM}{r^3}(x_2 - x_1) = -\frac{GM}{r^3}(X) = 6.371330 \, m/s^2$$

$$F_y = -\frac{GM(Y)}{r^3} = -4.466780 \, m/s^2$$

$$F_z = -\frac{GM(Z)}{r^3} = 5.995258 \, m/s^2$$

and the magnitude of the gravitational force F

$$F = 9.822\,886 \; m/s^2$$

Thus the GRAVITATIONAL FORCE $\underset{\sim}{F}$ acting on a unit mass at point P located a distance r from mass m is

$$\underset{\sim}{F} = -\frac{Gm}{r^3}\left\{(x_p - x)\underset{\sim}{i} + (y_p - y)\underset{\sim}{j} + (z_p - z)\underset{\sim}{k}\right\}$$

$$= -\frac{Gm}{r^3}\underset{\sim}{r} \qquad \text{.... (8)}$$

THE GRAVITY FIELD OF THE EARTH

THE GRAVITATIONAL POTENTIAL

Let

$$\boxed{V = \frac{Gm}{r}} \qquad \text{.... (9)}$$

be a SCALAR FUNCTION called the GRAVITATIONAL POTENTIAL between the mass m and a unit mass at P a distance r from m.

Differentiating (9) with respect to x_p gives

$$\frac{\partial V}{\partial x_p} = \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial x_p} = -\frac{Gm}{r^2}\frac{\partial r}{\partial x_p}$$

and since $r^2 = (x_p - x)^2 + (y_p - y)^2 + (z_p - z)^2$

then $2r\,\partial r = 2(x_p - x)\,\partial x_p$

$$\frac{\partial r}{\partial x_p} = \frac{(x_p - x)}{r}$$

giving

$$\boxed{\frac{\partial V}{\partial x_p} = -\frac{Gm}{r^3}(x_p - x)} \qquad \text{..... (10 a)}$$

and similarly.

$$\boxed{\frac{\partial V}{\partial y_p} = -\frac{Gm}{r^3}(y_p - y)} \qquad \text{... (10 b)}$$

$$\boxed{\frac{\partial V}{\partial z_p} = -\frac{Gm}{r^3}(z_p - z)} \qquad \text{.... (10 c)}$$

THE GRAVITY FIELD OF THE EARTH

Inspection of eqn's (2), (7) and (10) shows that the derivatives of the potential V are the components of the gravitational force $\underset{\sim}{F}$ and

$$\underset{\sim}{F} = \frac{\partial V}{\partial x}\underset{\sim}{i} + \frac{\partial V}{\partial y}\underset{\sim}{j} + \frac{\partial V}{\partial z}\underset{\sim}{k} \qquad \ldots (11)$$

or

$$\boxed{\underset{\sim}{F} = \text{grad } V = \nabla V} \qquad \ldots (12)$$

where "grad" is the GRADIENT which is represented by the vector differential operator ∇ (del or nabla) where

$$\nabla = \frac{\partial}{\partial x}\underset{\sim}{i} + \frac{\partial}{\partial y}\underset{\sim}{j} + \frac{\partial}{\partial z}\underset{\sim}{k} \qquad \ldots (13)$$

Equation (12) is of basic importance and shows that the three components of the vector $\underset{\sim}{F}$ can be replaced by a single function V (the potential).

In geodesy, the Earth can be considered as a solid body consisting of a summation of point masses thus the potential of the Earth is a summation of the potentials of point masses

$$V = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} + \ldots\ldots + \frac{Gm_n}{r_n} = G\sum_{i=1}^{n}\frac{m_i}{r_i} \qquad \ldots (14)$$

and assuming the point masses $m_1, m_2, \ldots m_n$ are distributed continuously over a volume v with a density ρ

$$\rho = \frac{dm}{dv} \qquad \ldots (15)$$

THE GRAVITY FIELD OF THE EARTH

where dm is an element of mass
dv " " element of volume

and density $\rho = \frac{\text{mass}}{\text{volume}}$

Then the summation (14) becomes an integral

$$V = G \iiint_{\text{earth}} \frac{dm}{r} = G \iiint \frac{\rho}{r} dv \qquad \text{.... (16)}$$

Note that the element of volume $dv = dx\,dy\,dz$ which explains the reason for the triple integral in (16).

Now, integral theorems of Gauss show that

$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{n}}\, dS = \begin{cases} 0 & \text{at all points outside Surface S containing all matter} \\ -4\pi G \iiint_V \rho\, dv & \text{at all points inside Surface S containing all matter.} \end{cases}$$

.... (17)

S is a surface containing a volume V
$\underset{\sim}{F}$ is gravitational force
$\underset{\sim}{\hat{n}}$ is unit normal to surface S
dS is element of surface area
ρ is density of matter
dv is element of volume
G is universal constant of gravitation

THE GRAVITY FIELD OF THE EARTH

Now since $\underset{\sim}{F} = \nabla V$ (see eq'n 12)

$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{n}}\, dS = \iint_S \nabla V \cdot \underset{\sim}{\hat{n}}\, ds$$

$$= \iiint_v \nabla \cdot (\nabla V)\, dv \qquad \text{(divergence theorem)}$$

$$= \iiint_v \nabla^2 V\, dv \qquad \cdots\cdots (18)$$

Noting that the "divergence theorem" (also known as Gauss' divergence theorem) states that for a closed surface S bounding a volume v and $\underset{\sim}{F}$ is a vector function having continuous partial derivatives in the region then

$$\iiint_v \nabla \cdot \underset{\sim}{F}\, dv = \iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{n}}\, dS$$

$\hat{n}$ is a unit normal

$\nabla \cdot \underset{\sim}{F}$ is the vector dot product

$$\nabla \cdot F = \left(\frac{\partial}{\partial x} \underset{\sim}{i} + \frac{\partial}{\partial y} \underset{\sim}{j} + \frac{\partial}{\partial z} \underset{\sim}{k} \right) \cdot \left(F_x \underset{\sim}{i} + F_y \underset{\sim}{j} + F_z \underset{\sim}{k} \right)$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Now, since $\iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{n}}\, dS = \iiint_v \nabla^2 V\, dv$ (see (18) above)

then from (17)

$$\iiint_v \nabla^2 V\, dv = \begin{cases} 0 & \text{at all points outside surface S containing all matter} \\ -4\pi G \iiint_v \rho\, dv & \text{at all points inside surface containing all matter} \end{cases}$$

THE GRAVITY FIELD OF THE EARTH

This leads to two important equations of physical geogesy involving the potential

(i) LAPLACE'S EQUATION

$$\boxed{\nabla^2 V = 0} \qquad \cdots (19)$$

true for all points outside the surface containing all matter.

(ii) POISSON'S EQUATION

$$\boxed{\nabla^2 V = -4\pi G \rho} \qquad \cdots (20)$$

true for all points inside the surface containing all matter.

NOTE $$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}} \qquad \cdots (21)$$

and in some geodesy texts $\nabla^2 V = \Delta V$

NOTE Laplace's eqn can be solved to give expressions for the Earth's potential.

THE GRAVITY FIELD OF THE EARTH.

CENTRIFUGAL FORCE & CENTRIFUGAL POTENTIAL

Points on the surface of the Earth (rotating at an angular velocity ω) are subject to a CENTRIFUGAL FORCE

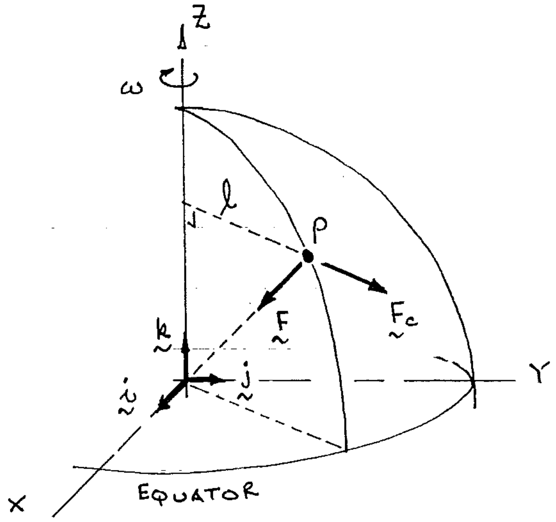


Fig 5. Centrifugal Force

$$\underset{\sim}{F_c} = m\omega^2 \underset{\sim}{\ell} \qquad \ldots (22)$$

m = mass

ω = angular velocity

$\underset{\sim}{\ell} = x_p \underset{\sim}{i} + y_p \underset{\sim}{j} + z_p \underset{\sim}{k}$

$\ell = \sqrt{x_p^2 + y_p^2}$

For a unit mass, the centrifigual force $\underset{\sim}{F_c}$ is

$$\underset{\sim}{F_c} = \omega^2 (x_p \underset{\sim}{i} + y_p \underset{\sim}{j}) \qquad \ldots (23)$$

The POTENTIAL of this centrifugal force is

$$Q = \frac{1}{2}\omega^2 (x^2 + y^2) \qquad \ldots (24)$$

So that

$$\underset{\sim}{F_c} = \nabla Q = \frac{\partial Q}{\partial x}\underset{\sim}{i} + \frac{\partial Q}{\partial y}\underset{\sim}{j} + \frac{\partial Q}{\partial z}\underset{\sim}{k} \qquad \ldots (25)$$

$$= \omega^2 x_p \underset{\sim}{i} + \omega^2 y_p \underset{\sim}{j} + 0$$

$$= \omega^2 (x_p \underset{\sim}{i} + y_p \underset{\sim}{j})$$

THE GRAVITY FIELD OF THE EARTH

Also $$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 2\omega^2 \qquad \ldots\ldots (26)$$

noting $$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) = \omega^2$$

$$\frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial y}\right) = \omega^2$$

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z}\right) = 0$$

THE GRAVITY POTENTIAL OF THE EARTH

The Earth's GRAVITY POTENTIAL is the sum of its GRAVITATIONAL and CENTRIFUGAL (or ROTATIONAL) potentials

$$W = V + \Phi \qquad \ldots\ldots (27)$$

gravity potential → W; gravitational potential → V; Rotational potential → Φ

where the CENTRIFUGAL (or ROTATIONAL) potential Φ can be represented by a simple analytical expression (see eq. 24)

$$\Phi = \frac{1}{2}\,\omega^2 (x^2 + y^2)$$

but the GRAVITATIONAL potential V cannot (see eq. 16)

$$V = G \iiint_v \frac{\rho}{r}\, dv$$

since ρ (the density of the Earth) is not completely and hence ρ is not a continuous variable.

If we restrict ourselves to the exterior space (outside the Earth's surface) and disregard the mass of the Earth's atmosphere we can consider this space to be mass free, ie. $\rho = 0$

In this region, LAPLACE'S EQU'N is satisfied

$$\nabla^2 V = 0 \qquad \text{(Laplace's equ'n)}$$

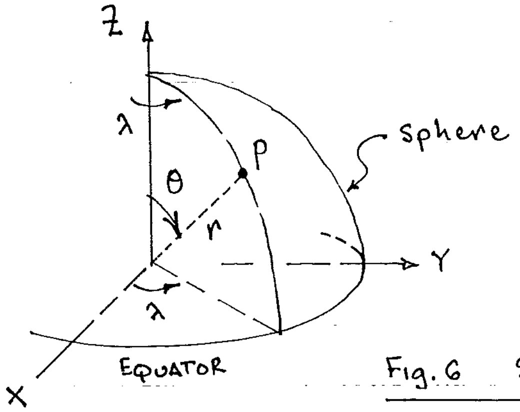
and a solution for V (the GRAVITATIONAL POTENTIAL) can be obtained using a SPHERICAL HARMONIC EXPANSION.

LAPLACE'S EQU'N in CARTESIAN COORDS. (x, y, z)

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0} \qquad \text{..... (28)}$$

LAPLACE'S EQU'N in SPHERICAL COORDINATES (r, θ, λ)

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \lambda^2} = 0} \qquad \text{.... (29)}$$



r = radius of sphere
θ = colatitude
λ = longitude

Fig. 6 Spherical coords r, θ, λ

THE GRAVITY FIELD OF THE EARTH

SOLUTION OF LAPLACE'S EQU'N

To find a solution of (29) first multiply (29) by $r^2 \sin\theta$ to give

$$\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \lambda}\left(\frac{1}{\sin\theta}\frac{\partial V}{\partial \lambda}\right) = 0$$

and differentiating and expanding gives

$$r^2 \sin\theta \frac{\partial^2 V}{\partial r^2} + 2r \sin\theta \frac{\partial V}{\partial r} + \sin\theta \frac{\partial^2 V}{\partial \theta^2} + \cos\theta \frac{\partial V}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial^2 V}{\partial \lambda^2} = 0 \quad \text{.... (30)}$$

Let $V(r,\theta,\lambda)$ be a solution of (30) and furthermore, consider the solution to be a product of two functions of the form

$$V(r,\theta,\lambda) = R(r)\, S(\theta,\lambda) \quad \text{.... (31)}$$

where $R(r)$ is a function of the radius r only
$S(\theta,\lambda)$ is a function of the parameters θ, λ
(θ, λ are surface parameters)

Substituting (31) into (30) gives

$$r^2 \sin\theta \frac{\partial^2}{\partial r^2}\left\{R(r)\, S(\theta,\lambda)\right\} + 2r \sin\theta \frac{\partial}{\partial r}\left\{R(r)\, S(\theta,\lambda)\right\}$$

$$+ \sin\theta \frac{\partial^2}{\partial \theta^2}\left\{R(r)\, S(\theta,\lambda)\right\} + \cos\theta \frac{\partial}{\partial \theta}\left\{R(r)\, S(\theta,\lambda)\right\}$$

$$+ \frac{1}{\sin\theta}\frac{\partial^2}{\partial \lambda^2}\left\{R(r)\, S(\theta,\lambda)\right\} = 0$$

THE GRAVITY FIELD OF THE EARTH

Noting that $\frac{\partial}{\partial r}\left(S(\theta,\lambda)\right) = 0$, $\frac{\partial}{\partial \theta}\left(R(r)\right) = 0$, etc....

then

$$r^2 \sin\theta \cdot S(\theta,\lambda) \cdot \frac{\partial^2 R}{\partial r^2} + 2r \sin\theta \cdot S(,\lambda) \cdot \frac{\partial R}{\partial r} + \sin\theta \cdot R(r) \cdot \frac{\partial^2 S}{\partial \theta^2}$$

$$+ \cos\theta \cdot R(r) \cdot \frac{\partial S}{\partial \theta} + \frac{1}{\sin\theta} \cdot R(r) \cdot \frac{\partial^2 S}{\partial \lambda^2} = 0$$

and dividing both sides by $\sin\theta$ and gathering terms gives

$$S(\theta,\lambda)\left\{ r^2 \frac{\partial^2 R}{\partial r^2} + 2r \frac{\partial R}{\partial r} \right\} + R(r)\left\{ \frac{\partial^2 S}{\partial \theta^2} + \cot\theta \frac{\partial S}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 S}{\partial \lambda^2} \right\} = 0$$

.... (32)

The two 2^{nd} order differential equations (DE's) in (32) have to be satisfied simultaneously and the two equations, one in terms of r, the other in terms of θ and λ, may be solved by equating both to the same constant. Letting this constant equal $n(n+1)$ enables a simple solution for the 2^{nd} order D.E. in r. Hence, re-arranging (32) and letting equal to $n(n+1)$ gives

$$\frac{1}{R(r)}\left\{ r^2 \frac{\partial^2 R}{\partial r^2} + 2r \frac{\partial R}{\partial r} \right\} = n(n+1)$$

$$-\frac{1}{S(\theta,\lambda)}\left\{ \frac{\partial^2 S}{\partial \theta^2} + \cot\theta \frac{\partial S}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 S}{\partial \lambda^2} \right\} = n(n+1)$$

THE GRAVITY FIELD OF THE EARTH

and $$\boxed{r^2 \frac{\partial^2 R}{\partial r^2} + 2r \frac{\partial R}{\partial r} - n(n+1) R(r) = 0} \qquad \ldots\ldots (33)$$

$$\frac{\partial^2 S}{\partial \theta^2} + \cot\theta \frac{\partial S}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 S}{\partial \lambda^2} + n(n+1) \cdot S(\theta,\lambda) = 0 \qquad \ldots\ldots (34)$$

Now, to find a solution for (34), the function $S(\theta,\lambda)$ can be considered as the product of two functions $F(\lambda)$ and $G(\theta)$ – functions of λ and θ respectively.

ie, $$S(\theta,\lambda) = F(\lambda) \cdot G(\theta) \qquad \ldots\ldots (35)$$

Substituting (35) into (34) gives

$$F(\lambda) \frac{\partial^2 G}{\partial \theta^2} + \cot\theta \cdot F(\lambda) \cdot \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2\theta} \cdot G(\theta) \frac{\partial^2 F}{\partial \lambda^2} + n(n+1) \cdot F(\lambda) \cdot G(\theta) = 0$$

Multiplying both sides by $\sin^2\theta$ and re-arranging gives

$$F(\lambda) \left\{ \sin^2\theta \frac{\partial^2 G}{\partial \theta^2} + \sin\theta \cos\theta \frac{\partial G}{\partial \theta} + n(n+1) \sin^2\theta \cdot G(\theta) \right\} + G(\theta) \frac{\partial^2 F}{\partial \lambda^2} = 0$$

or

$$\frac{1}{G(\theta)} \left\{ \sin^2\theta \frac{\partial^2 G}{\partial \theta^2} + \sin\theta \cos\theta \frac{\partial G}{\partial \theta} + n(n+1) \sin^2\theta \cdot G(\theta) \right\} = -\frac{1}{F(\lambda)} \frac{\partial^2 F}{\partial \lambda^2}$$

$\ldots\ldots (36)$

Now, similarly to before, the two D.E.'s in (36) must be satisfied simultaneously which can be achieved by equating both to the same constant m^2 giving

$$\frac{1}{G(\theta)} \left\{ \sin^2 \frac{\partial^2 G}{\partial \theta^2} + \sin\theta \cos\theta \frac{\partial G}{\partial \theta} + n(n+1) \sin^2\theta \cdot G(\theta) \right\} = m^2$$

and $$-\frac{1}{F(\lambda)} \frac{\partial^2 F}{\partial \lambda^2} = m^2$$

giving

$$\boxed{\frac{\partial^2 F}{\partial \lambda^2} + m^2 F(\lambda) = 0} \quad \text{....(37)}$$

and

$$\sin^2\theta \frac{\partial^2 G}{\partial \theta^2} + \sin\theta \cos\theta \frac{\partial G}{\partial \theta} + \left\{ n(n+1)\sin^2\theta - m^2 \right\} G(\theta) = 0 \quad \text{....(38)}$$

The D.E. (38) can be reduced to a "standard form" by the substitution

$$t = \cos\theta \quad \text{... (39)}$$

giving

$$\frac{\partial t}{\partial \theta} = -\sin\theta \quad \text{.... (40 a)}$$

and

$$\sin^2\theta = 1 - \cos^2\theta = 1 - t^2 \quad \text{.... (40 b)}$$

Using the chain rule for differentiation, (38) can be simplified as follows

$$\frac{\partial G}{\partial \theta} = \frac{\partial G}{\partial t} \cdot \frac{\partial t}{\partial \theta} = -\sin\theta \frac{\partial G}{\partial t} \quad \text{.... (41 a)}$$

$$\frac{\partial^2 G}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial G}{\partial \theta}\right) = -\sin\theta \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial t}\right)\frac{\partial t}{\partial \theta} - \cos\theta \frac{\partial G}{\partial t}$$

$$= -\sin\theta \frac{\partial^2 G}{\partial t^2} \cdot \frac{\partial t}{\partial \theta} - \cos\theta \frac{\partial G}{\partial t}$$

$$= (1-t^2)\frac{\partial^2 G}{\partial t^2} - t\frac{\partial G}{\partial t} \quad \text{.... (41 b)}$$

and substituting (41) and (40) into (38) gives

$$(1-t^2)\left\{(1-t^2)\frac{\partial^2 G}{\partial t^2} - t\frac{\partial G}{\partial t}\right\} - (1-t^2)\, t\frac{\partial G}{\partial t} + (1-t^2)\left\{ n(n+1) - \frac{m^2}{1-t^2} \right\} G(t) = 0$$

where $G(\theta) \sim G(t)$ since $t = \cos\theta$

THE GRAVITY FIELD OF THE EARTH

dividing both sides by $1-t^2$ gives

$$\boxed{(1-t^2)\frac{\partial^2 G}{\partial t^2} - 2t\frac{\partial G}{\partial t} + \left\{n(n+1) - \frac{m^2}{1-t^2}\right\} G(t) = 0} \quad \ldots\ldots(42)$$

where $t = \cos\theta$

Eqn (42) is known as LEGENDRE'S ASSOCIATED DIFFERENTIAL EQUATION and holds only if $t \neq \pm 1$ (ie. $\theta \neq 0°$ or $180°$)

A solution of Legendre's associated differential equation is given by

$$G = P_n^m(t) \quad \ldots (43)$$

where $P_n^m(t)$ is called an ASSOCIATED LEGENDRE FUNCTION of degree n and order m where n, m are zero or positive integers and $P_n^m(t) = 0$ for $m > n$

When $m = 0$, (42) becomes

$$\boxed{(1-t^2)\frac{\partial^2 G}{\partial t^2} - 2t\frac{\partial G}{\partial t} + n(n+1)\,G(t) = 0} \quad \ldots (44)$$

which is known as LEGENDRE'S DIFFERENTIAL EQUATION, a solution of which is given by

$$G = P_n(t) \quad \ldots (45)$$

where $P_n(t)$ is called a LEGENDRE POLYNOMIAL

THE GRAVITY FIELD OF THE EARTH

Legendre functions will be considered in detail in the following sections

Now, from (31) and (35)

$$V = R(r) \cdot S(\theta, \lambda)$$
$$= R(r) \cdot F(\lambda) \cdot G(t) \qquad \text{where } t = \cos\theta$$

and standard 2nd order differential equations for R, F and G are given by :

$$\boxed{r^2 \frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} - n(n+1)R = 0} \qquad \cdots\ (33)$$

$$\boxed{\frac{\partial^2 F}{\partial \lambda^2} + m^2 F = 0} \qquad \cdots\ (37)$$

$$\boxed{(1-t^2)\frac{\partial^2 G}{\partial t^2} - 2t\frac{\partial G}{\partial t} + \left\{ n(n+1) - \frac{m^2}{1-t^2} \right\} G = 0} \qquad \cdots\ (42)$$

Standard solutions for these equations are

(i) Solutions for R

(a) $R = Ar^n \qquad \cdots\ (46a)$

(b) $R = \dfrac{B}{r^{(n+1)}} \qquad \cdots\ (46b)$

(c) $R = Ar^n + \dfrac{B}{r^{(n+1)}} \qquad \cdots\ (46c)$

THE GRAVITY FIELD OF THE EARTH

(ii) Solutions for F

(a) $F = C\cos(m\lambda)$ .... (47 a)

(b) $F = D\sin(m\lambda)$ .... (47 b)

(c) $F = C\cos(m\lambda) + D\sin(m\lambda)$ .... (47 c)

(iii) Solutions for G

$$G = P_n^m(t) \quad \text{where } t = \cos\theta \qquad \text{.... (43)}$$

Equations (46) are the three possible solutions for R with constants A and B having different values for different n. Since we expect the gravitational potential of a body to decrease with increasing distance from the body, then the solution $R = A r^n$ (being a part of the solution for V) would mean V increasing with increasing distance. This solution for R is therefore inadmissable.

Since there are an infinite number of solutions to (46b) $n = 0, 1, 2, \dots \infty$ and each is a solution to (33) then the sum is also a solution, hence

$$R(r) = \sum_{n=0}^{\infty} \left(\frac{B_n}{r^{(n+1)}} \right) \qquad \text{.... (48)}$$

Using similar reasoning, a solution to (37) is

$$F(\lambda) = \sum_{m=0}^{\infty} \left\{ C_m \cos(m\lambda) + D_m \sin(m\lambda) \right\} \qquad \text{.... (49)}$$

THE GRAVITY FIELD OF THE EARTH

Combining (43), (48) and (49) as per (31) and (35) gives the solution for the external gravitational potential as

$$\boxed{V = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} P_n^m(t) \left\{ a_n^m \cos(m\lambda) + b_n^m \sin(m\lambda) \right\}} \quad \text{--- (50)}$$

where a_n^m and b_n^m are arbitrary constants.

(see Heiskanen & Moritz (1967), p.21, eq 1.54b)

Eq. (50) is a solution of Laplace's eqn $\nabla^2 V = 0$ and is known as a <u>SPHERICAL HARMONIC SERIES</u> and such series expansions are extremely useful in geodesy where they can be used to develop <u>GEOPOTENTIAL MODELS</u> of the Earth.

By considering r = unity, the potential $V(r,\theta,\lambda)$ can be represented as an arbitrary function $f(\theta,\lambda)$ on the surface of a unit sphere. Such a function can be expanded into a series of <u>SURFACE SPHERICAL HARMONICS</u>

$$f(\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(t) \left\{ a_n^m \cos(m\lambda) + b_n^m \sin(m\lambda) \right\} \quad \text{--- (51)}$$

THE GRAVITY FIELD OF THE EARTH

LEGENDRE FUNCTIONS AND POLYNOMIALS $P_n^m(t)$, $P_n(t)$

The associated Legendre function of a variable $t = \cos\theta$, $P_n^m(t)$ are defined by

$$\boxed{P_n^m(t) = \frac{1}{2^n n!}(1-t^2)^{m/2}\frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n} \quad \text{---- (52)}$$

and $P_n^m(t) = 0$ for $m > n$

Example: $n=1$, $m=1$

$$P_1^1(t) = \frac{1}{2\cdot 1}(1-t^2)^{1/2}\frac{d^2}{dt^2}(t^2-1)$$

$$= \frac{1}{2}\sqrt{1-t^2}\cdot 2$$

$$\underline{P_1^1(t) = \sqrt{1-t^2} = \sin\theta}$$

Example: $n=3$, $m=2$

$$P_3^2(t) = \frac{1}{2^3\, 3!}(1-t^2)^1\frac{d^5}{dt^5}(t^2-1)^3$$

$$= \frac{1}{48}(1-t^2)\cdot 720\,t$$

$$\underline{P_3^2(t) = 15\,t\,(1-t^2) = 15\cos\theta\sin^2\theta}$$

and

$$\begin{aligned}
P_1^1(t) &= (1-t^2)^{1/2} &&= \sin\theta \\
P_2^0(t) &= \tfrac{1}{2}(3t^2-1) &&= \tfrac{1}{4}(1+3\cos 2\theta) \\
P_2^1(t) &= 3t(1-t^2)^{1/2} &&= 3\cos\theta\sin\theta \\
P_2^2(t) &= 3(1-t^2) &&= 3\sin^2\theta \\
&\vdots
\end{aligned}$$

The special case, when $m=0$ give rise to Legendre polynomials

$$\boxed{P_n^0(t) = P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n} \qquad \cdots (53)$$

and for $t = \cos\theta$, $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$

$$\begin{aligned}
P_0(t) &= 1 \\
P_1(t) &= t &&= \cos\theta \\
P_2(t) &= \tfrac{1}{2}(3t^2-1) &&= \tfrac{1}{4}(1+3\cos 2\theta) \\
P_3(t) &= \tfrac{1}{2}(5t^3-3t) &&= \tfrac{1}{8}(3\cos\theta + 5\cos 3\theta)
\end{aligned}$$

Noting

$$\begin{aligned}
P_0^0(t) &= P_0(t) \\
P_1^0(t) &= P_1(t) \\
P_2^0(t) &= P_2(t) \\
&\vdots
\end{aligned}$$

ie, when $m=0$ the associated Legendre function is the Legendre polynomial

Inspection of equations (52) and (53) show the associated Legendre functions are related to Legendre polynomials by.

$$\boxed{P_n^m(t) = (1-t^2)^{m/2}\frac{d^m}{dt^m}P_n(t)} \qquad \cdots\cdot (54)$$

THE GRAVITY FIELD OF THE EARTH

In practical applications, Legendre functions are computed by RECURRENCE FORMULAE. The following are four fundamental relations from which others may be derived

$$(2n+1)\,t\,P_n^m = (n+m)P_{n-1}^m + (n-m+1)P_{n+1}^m \quad \dots\ (55a)$$

$$(2n+1)\,u\,P_n^m = P_{n+1}^{m+1} - P_{n-1}^{m+1} \quad \dots\ (55b)$$

$$(2n+1)\,u\,P_n^m = (n+m)(n+m-1)P_{n-1}^{m-1} - (n-m+1)(n-m+2)P_{n+1}^{m-1} \quad \dots (55c)$$

$$(2n+1)\,u^2 \frac{\partial}{\partial t} P_n^m = (n+1)(n+m)P_{n-1}^m - n(n-m+1)P_{n+1}^m \quad \dots (55d)$$

where

$$t = \cos\theta \qquad \text{and} \qquad P_n^m = P_n^m(t)$$
$$u^2 = 1 - t^2$$

The recurrence formulae above, can be found in Wang, Z.X. and Guo, D.R. (1989) "Special Functions", World Scientific Publishing Co., Singapore, pp. 239-241, but with the following changes

Since the definition of the Legendre function used by Wang & Guo (and other reference texts) includes a coefficient $(-1)^m$

$$P_n^m(t) = (-1)^m (1-t^2)^{m/2} \frac{d^m}{dt^m} P_n(t)$$

and the definition of $P_n^m(t)$ used in these notes and other geodesy texts does not (see eq. 54) then recurrence formula may differ by a coefficient of (-1) when terms of varying order are involved.

THE GRAVITY FIELD OF THE EARTH

COMPUTATION OF LEGENDRE FUNCTIONS

Consider the computation of Legendre functions to degree and order five (5) for a particular point

In symbolic form, the Legendre functions $P_n^m(t)$ or simply P_n^m can be arranged as

order

degree $n \backslash m$	0	1	2	3	4	5
0	P_0^0					
1	P_1^0	P_1^1				
2	P_2^0	P_2^1	P_2^2			
3	P_3^0	P_3^1	P_3^2	P_3^3		
4	P_4^0	P_4^1	P_4^2	P_4^3	P_4^4	
5	P_5^0	P_5^1	P_5^2	P_5^3	P_5^4	P_5^5

$P_n^m = 0$ for $n > m$

Fig. 7 Legendre Functions

From (55 a) replacing n with $n-1$ gives

$$(2n-1)\, t\, P_{n-1}^m = (n+m-1)\, P_{n-2}^m + (n-m)\, P_n^m$$

giving

$$\boxed{P_n^m = \frac{2n-1}{n-m}\, t\, P_{n-1}^m - \frac{n+m-1}{n-m}\, P_{n-2}^m} \qquad \text{.... (56 a)}$$

a recurrence formula for computing P_n^m when $n \neq m$ providing P_{n-1}^m and P_{n-2}^m are known.

For $m = 0$, $P_0^0 = 1$ and $P_1^0 = t$ can be used as "seeds" to start the calculation of P_n^m in column $m = 0$.

The diagonal Legendre functions P_m^m may be computed by considering (55b) with $m-1$ replacing m giving

$$(2m-1)\,u\,P_n^{m-1} = P_{n+1}^{m} - P_{n-1}^{m}$$

replacing n with $m-1$ gives

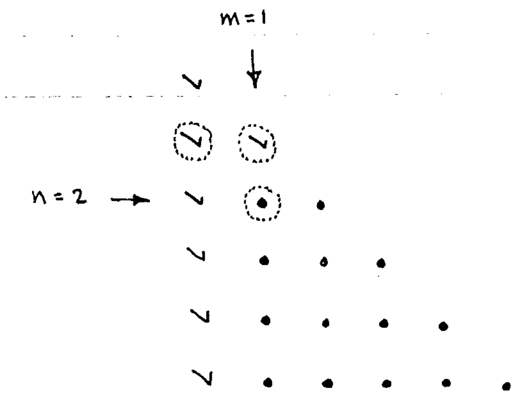
$$(2m-1)\,u\,P_{m-1}^{m-1} = P_{m}^{m} - P_{m-2}^{m}$$

and since the order will be always greater than the degree in the last term on the right hand side of the equation; it is zero by definition ($P_n^m = 0$ when $m > n$) hence

$$\boxed{P_m^m = (2m-1)\,u\,P_{m-1}^{m-1}} \qquad \text{.... (56b)}$$

Now, using (56b) the first function in the next column (the diagonal element) can be computed.

In schematic form, the computed Legendre functions are show thus ($\checkmark$) and the unknown functions ($\cdot$)



To compute the functions in the next "column" ($m=1$) the first two values must be known, but only the diagonal element is known at this stage.

A recurrence relationship can be established linking two of the values shown above; two are already known and one is unknown

THE GRAVITY FIELD OF THE EARTH

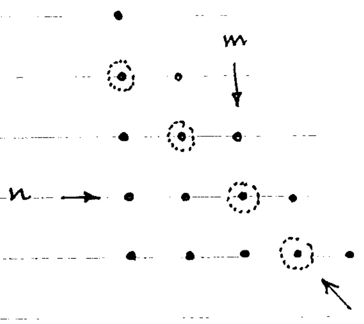
From (55b) with m-1 replacing m

$$(2n+1)\,u\,P_n^{m-1} = P_{n+1}^{m} - P_{n-1}^{m}$$

and replacing n with n-1 gives

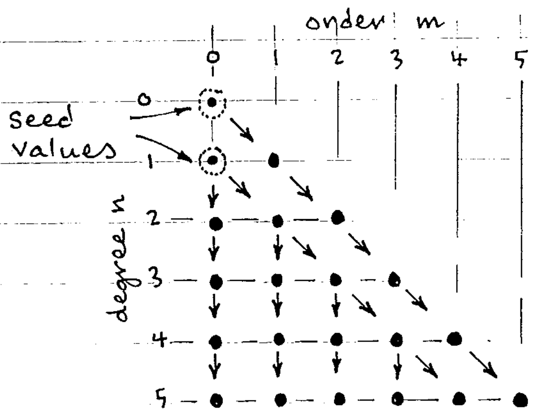
$$(2n-1)\,u\,P_{n-1}^{m-1} = P_n^{m} - P_{n-2}^{m}$$

Now, if this relationship is used to compute elements in the "2nd-diagonal" only, ie the elements shown below, then "m" will always be one less than "n" and the last term on the right-hand-side above will be zero, since $P_n^m = 0$ when $m > n$, hence



$$\boxed{P_n^m = (2n-1)\,u\,P_{n-1}^{m-1}} \qquad \text{for } n = m+1 \qquad \ldots(56\text{c})$$

(2nd diagonal only)



- Diagonal elements P_m^m computed using (56 b)
- 2nd diagonal elements computed using (56c)
- Other elements computed using (56a)

THE GRAVITY FIELD OF THE EARTH

For Melbourne ϕ: $-37°48'00''$ λ: E $144°58'00''$, $h = 0.000$ m

on the GRS80 ellipsoid ($a = 6378137.000$ m, $f = 1/298.257222100881$)

the geocentric Cartesian coords are

$X = -4131810.563$ m.

$Y = 2896708.708$ m.

$Z = -3887927.165$

$r = 6370145.800$

$\theta = 127°\ 36'\ 49.5994$ geocentric co-latitude

$\psi = -37°\ 36'\ 49.5994$ " latitude.

$t = \cos\theta = -0.610335664$

$u = \sin\theta = 0.792142902$

The associated Legendre functions $P_n^m(t)$ to degree & order 4

ORDER →

DEGREE ↓

	m = 0	m = 1	m = 2	m = 3	m = 4
n = 0	1				
n = 1	−0.610335664	0.792142902			
n = 2	0.058764434	−1.450419192	1.882471132		
n = 3	0.347113726	1.024892048	−5.744696341	7.455930728	
n = 4	−0.414821127	0.474326529	7.565497864	−31.854343018	41.343138233

check values

$P_2^1(t) = 3ut$

$P_2^2(t) = 3u^2$

$P_3^0(t) = t/2\,(5t^2-3)$

$P_3^1(t) = \frac{3}{2}u(5t^2-1)$

$P_3^2(t) = 15tu^2$

$P_3^3(t) = 15u^3$

$P_4^0(t) = \frac{1}{8}(35t^4 - 30t^2 + 3)$

$P_4^1(t) = \frac{5}{2}(7t^3 - 3t)u$

$P_4^2(t) = \frac{15}{2}(7t^2 - 1)u^2$

$P_4^3(t) = 105\,t\,u^3$

$P_4^4(t) = 105\,u^4$

Check values from Table 3.1, p.48, Mather, R.S. (1971). "The analysis of the Earths Gravity Field", School of Surveying, University of New South Wales, Kensington, NSW.

THE GRAVITY FIELD OF THE EARTH

An explicit formula for any Legendre function (polynomial or associated function) is given in Heiskanen & Moritz (1967), p. 24, eq. 1-62 as

$$P_n^m(t) = \frac{(1-t^2)^{m/2}}{2^n} \sum_{k=0}^{r} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-m-2k)!} t^{n-m-2k} \qquad \ldots (57)$$

where

$$r = \begin{cases} \dfrac{n-m}{2} & (n-m) \text{ even} \\ \dfrac{n-m-1}{2} & (n-m) \text{ odd} \end{cases}$$

For example: $P_4^2(t)$ where $t = \cos\theta$, $\theta = 127°48'$

$n = 4$, $m = 2$, $r = 1$

$$P_4^2(t) = \frac{(1-t^2)}{2^4} \left\{ \frac{(-1)^0 8!\, t^2}{0!\, 4!\, 2!} + \frac{(-1)^1 6!\, t^0}{1!\, 3!\, 0!} \right\}$$

$$= \frac{(1-t^2)}{16} \left\{ \frac{40320t^2}{48} - \frac{720}{6} \right\}$$

$$= \frac{15(1-t^2)}{2} \left\{ 7t^2 - 1 \right\}$$

$$P_4^2(t) = 7.630\,675\,511$$

ORTHOGONALITY RELATIONS OF LEGENDRE FUNCTIONS

Consider the general surface spherical harmonic of degree n (see eqn 51)

$$f(\theta,\lambda) = \sum_{n=0}^{\infty} Y_n(\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left\{ a_n^m R_n^m(\theta,\lambda) + b_n^m S_n^m(\theta,\lambda) \right\} \quad \cdots (58)$$

where

$$R_n^m(\theta,\lambda) = P_n^m(t) \cos m\lambda \quad \cdots (59a)$$

$$S_n^m(\theta,\lambda) = P_n^m(t) \sin m\lambda \quad \cdots (59b)$$

$$(t = \cos\theta)$$

and $R_n^m(\theta,\lambda)$, $S_n^m(\theta,\lambda)$ can be considered as Legendre functions.

To calculate the coefficients a_n^m and b_n^m of (58) the ORTHOGONALITY RELATIONS are used. These remarkable relations mean that <u>the integral over the unit sphere (radius = 1) of the product of any two different functions R_n^m or S_n^m is zero.</u>

$$\left.\begin{aligned} \iint_\sigma R_n^m(\theta,\lambda)\, R_s^r(\theta,\lambda)\, d\sigma &= 0 \\ \iint_\sigma S_n^m(\theta,\lambda)\, S_s^r(\theta,\lambda)\, d\sigma &= 0 \end{aligned}\right\} \text{ if } s \neq n \text{ or } r \neq m \text{ or both}$$

$$\iint_\sigma R_n^m(\theta,\lambda)\, S_s^r(\theta,\lambda)\, d\sigma = 0 \quad \cdots (60)$$

THE GRAVITY FIELD OF THE EARTH

For the product of two equal functions R_n^m or S_n^m

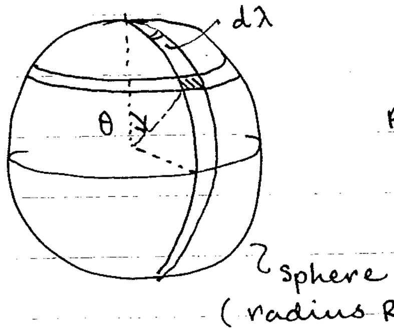
$$\iint_\sigma \left[R_n^0(\theta,\lambda)\right]^2 d\sigma = \frac{4\pi}{2n+1} \qquad \ldots (61)$$

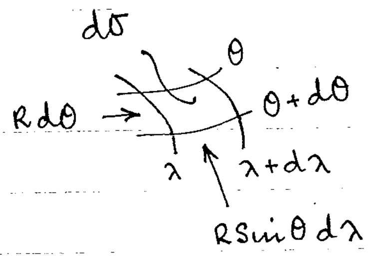
$$\iint_\sigma \left[R_n^m(\theta,\lambda)\right]^2 d\sigma = \iint_\sigma \left[S_n^m(\theta,\lambda)\right]^2 d\sigma = \frac{2\pi}{2n+1}\,\frac{(n+m)!}{(n-m)!}$$

Note that there is no S_n^0 since from (59b)

$S_n^0(\theta,\lambda) = P_n^0(t)\,\sin(0\lambda) = 0$

In formulas (60) and (61) the double integral $\iint_\sigma$ signifies an integration over ~~of~~ the surface of a sphere whose element of area $d\sigma = \sin\theta\, d\theta\, d\lambda$





$$\iint_\sigma d\sigma = \int_{\lambda=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta\, d\theta\, d\lambda$$

Now, to determine the coefficients a_n^m and b_n^m in (58) multiply both sides of the equation by a certain function $R_s^r(\theta,\lambda)$ and integrate over the unit sphere to give

$$\iint_\sigma f(\theta,\lambda) R_s^r(\theta,\lambda)\, d\sigma = a_s^r \iint \left[R_s^r(\theta,\lambda)\right]^2 d\sigma$$

noting that in the double integral on the right-hand side all terms except the one with $n=s$, $m=r$ will vanish according to the orthogonality relations. The integral on the right-hand side has the value given by (61) and similar expressions can be developed for b_s^r. The general result is

$$a_n^0 = \frac{2n+1}{4\pi} \iint_\sigma f(\theta,\lambda)\, P_n^m(t)\, d\sigma$$

$$a_n^m = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \iint_\sigma f(\theta,\lambda)\, R_n^m(\theta,\lambda)\, d\sigma \qquad \ldots\ldots (62)$$

$$b_n^m = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \iint_\sigma f(\theta,\lambda)\, S_n^m(\theta,\lambda)\, d\sigma$$

and the coefficients a_n^m, b_n^m can be determined by integration

THE GRAVITY FIELD OF THE EARTH

CONVENTIONAL AND FULLY NORMALIZED HARMONICS

For ease of computation in practical applications a FULLY NORMALIZED harmonic, denoted by $\bar{R}_n^m$ or $\bar{S}_n^m$ is defined such that its average squared value over the surface of a unit sphere is unity, ie

$$\frac{1}{4\pi}\iint_\sigma (\bar{R}_n^m)^2 \, d\sigma = \frac{1}{4\pi}\iint_\sigma (\bar{S}_n^m)^2 \, d\sigma = 1$$

(where the average = integral divided by area 4π)

This definition allows (58) to be written as

$$f(\theta,\lambda) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{\bar{a}_n^m \bar{R}_n^m(\theta,\lambda) + \bar{b}_n^m \bar{S}_n^m(\theta,\lambda)\right\} \qquad \dots (63)$$

where the fully normalized coefficients are

$$\bar{a}_n^m = \frac{1}{4\pi}\iint_\sigma f(\theta,\lambda)\,\bar{R}_n^m(\theta,\lambda)\, d\sigma$$

$$\bar{b}_n^m = \frac{1}{4\pi}\iint_\sigma f(\theta,\lambda)\,\bar{S}_n^m(\theta,\lambda)\, d\sigma \qquad \dots (64)$$

(noting that there is no separate formula required for $\bar{a}_n^0$ as there was when "conventional" harmonic functions R_n^m, S_n^m were used) and

$$\bar{R}_n^m(\theta,\lambda) = \bar{P}_n^m(t)\cos m\lambda$$

$$\bar{S}_n^m(\theta,\lambda) = \bar{P}_n^m(t)\sin m\lambda \qquad \dots (65)$$

THE GRAVITY FIELD OF THE EARTH

Conventional and Fully normalized harmonics are related by,

$$\left\{ \begin{matrix} \bar{R}_n^m(\theta,\lambda) \\ \bar{S}_n^m(\theta,\lambda) \end{matrix} \right\} = \sqrt{k(2n+1)\frac{(n-m)!}{(n+m)!}} \left\{ \begin{matrix} R_n^m(\theta,\lambda) \\ S_n^m(\theta,\lambda) \end{matrix} \right\} \qquad \text{.....(66)}$$

Conventional and Fully normalized Legendre functions are related by

$$\bar{P}_n^m(t) = \sqrt{k(2n+1)\frac{(n-m)!}{(n+m)!}}\; P_n^m(t) \qquad \text{....(67)}$$

and Conventional and Fully normalized coefficients are related by

$$\left\{ \begin{matrix} \bar{a}_n^m \\ \bar{b}_n^m \end{matrix} \right\} = \sqrt{\frac{(n+m)!}{k(2n+1)(n-m)!}} \left\{ \begin{matrix} a_n^m \\ b_n^m \end{matrix} \right\} \qquad \text{....(68)}$$

where $k = 1$ when $m = 0$

$k = 2$ when $m > 0$

Note also, that when $m = n$ then $(n-m)! = 0! = 1$

THE GRAVITY FIELD OF THE EARTH

SURFACE SPHERICAL HARMONICS - ZONAL, TESSERAL, SECTORIAL

The general surface spherical harmonic of degree n can be written as Y_n and contains $(2n+1)$ numerical terms (see eq. 51 and eq. 58)

$$Y_n = \sum_{m=0}^{n} P_n^m(t)\left(a_n^m \cos m\lambda + b_n^m \sin m\lambda\right) \quad \ldots (58)$$

and, as an example (Bomford 198, p.784) for $n=2$ there are 5 terms

$$Y_2 = P_2^0(t)\, a_2^0 + P_2^1(t)\left(a_2^1 \cos\lambda + b_2^1 \sin\lambda\right) + P_2^2(t)\left(a_2^2 \cos 2\lambda + b_2^2 \sin 2\lambda\right)$$

Single harmonic terms of degree n and order m is Y_n^m, for example

$$\begin{aligned}
Y_0^0 &= P_0^0(t)\, a_0^0 \\
Y_1^0 &= P_1^0(t)\, a_1^0 \\
Y_1^1 &= P_1^1(t)\left(a_1^1 \cos\lambda + b_1^1 \sin\lambda\right) \\
Y_2^0 &= P_2^0(t)\left(a_2^0\right) \\
Y_2^1 &= P_2^1(t)\left(a_2^1 \cos\lambda + b_2^1 \sin\lambda\right) \\
Y_2^2 &= P_2^2(t)\left(a_2^2 \cos 2\lambda + b_2^2 \sin 2\lambda\right) \\
Y_3^0 &= P_3^0(t)\left(a_3^0\right) \\
Y_3^1 &= P_3^1(t)\left(a_3^1 \cos\lambda + b_3^1 \sin\lambda\right) \\
Y_3^2 &= P_3^2(t)\left(a_3^2 \cos 2\lambda + b_3^2 \sin 2\lambda\right) \\
Y_3^3 &= P_3^3(t)\left(a_3^3 \cos 3\lambda + b_3^3 \sin 3\lambda\right)
\end{aligned}$$

THE GRAVITY FIELD OF THE EARTH

The geometric representation of SPHERICAL HARMONICS is important. Harmonics with $m=0$, ie

$$Y_0^0 = P_0^0 \, a_0^0 \qquad \text{where} \quad P_m^n \equiv P_n^m(t)$$
$$Y_1^0 = P_1^0 \, a_1^0 \qquad a_n^m = \text{coefficient}$$
$$Y_2^0 = P_2^0 \, a_2^0$$
$$Y_3^0 = P_3^0 \, a_3^0$$

will change their sign n times in the interval $0 \leq \theta \leq \pi$ and are independent of λ since there are no $\sin\lambda$ or $\cos\lambda$ terms in the harmonics. This has the effect of dividing the sphere into a number of zones and for this reason, harmonic terms where $m=0$ are known as ZONAL HARMONICS

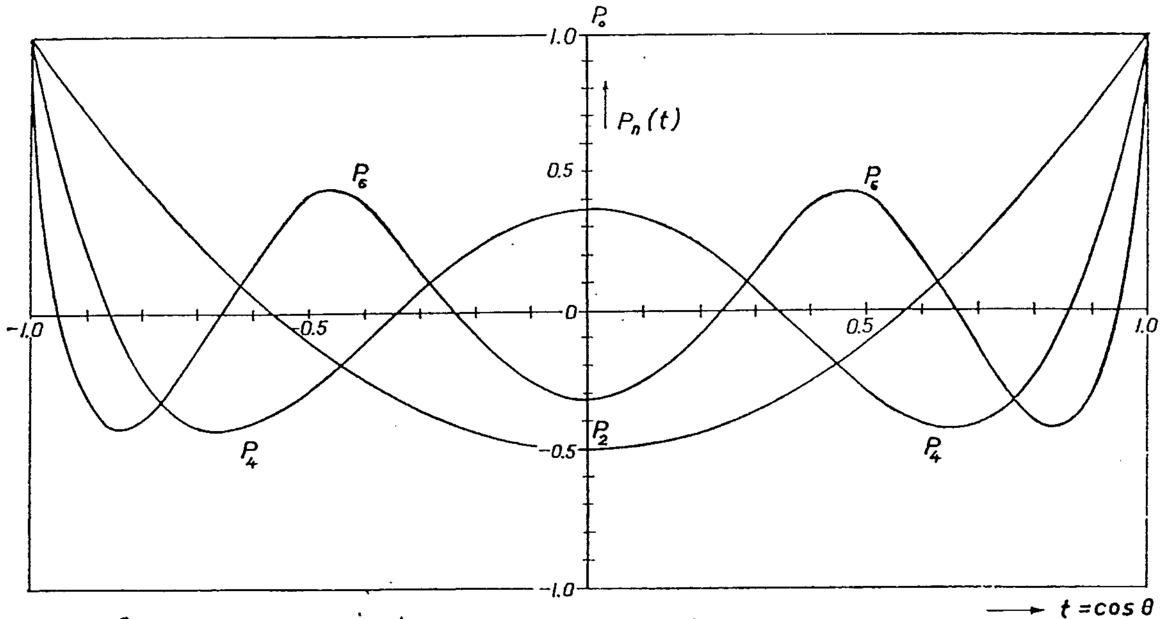
As an example, a 5^{th} degree zonal harmonic $(n=5)$

$$Y_5^0 = P_5^0 \, a_5^0 = \left(\frac{63}{128} \cos 5\theta + \frac{35}{128} \cos 3\theta + \frac{15}{64} \cos\theta \right) a_5^0$$

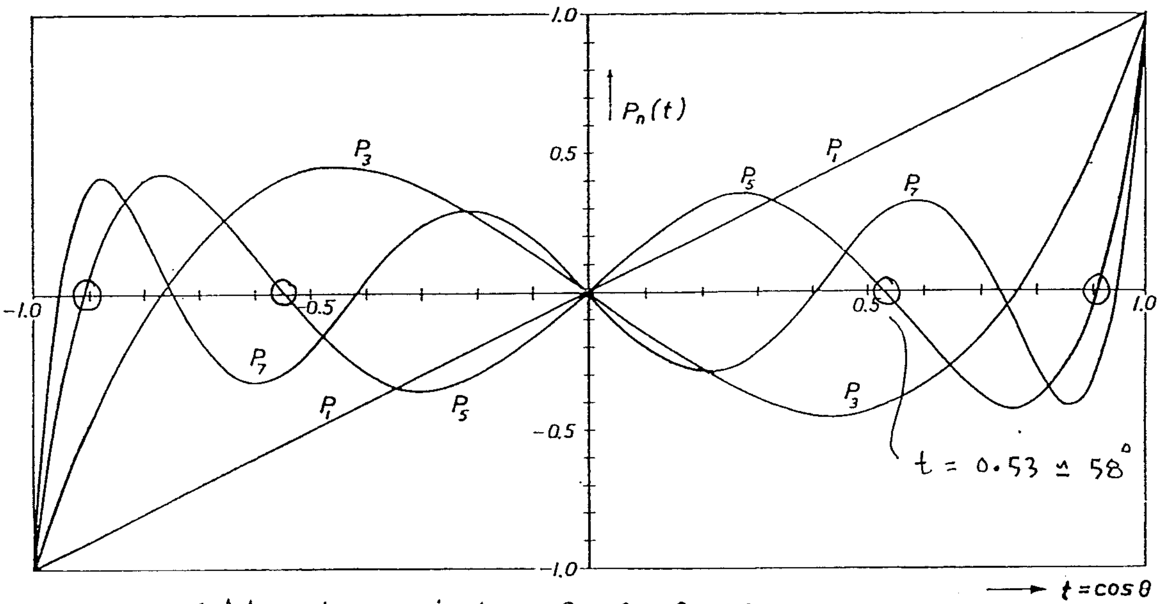
has zero's at (approx) $\theta = 155.5^\circ$
$= 122.0^\circ$
$= 90^\circ$
$= 58^\circ$
$= 24.5^\circ$

These values have been scaled from the diagram on p.39

THE GRAVITY FIELD OF THE EARTH



Even polynomials P_0, P_2, P_4, P_6.



Odd polynomials P_1, P_3, P_5, P_7

Legendre's polynomials as functions of $t = \cos\theta$. Top, n even; bottom, n odd.

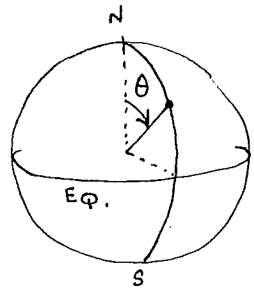
From the diagrams, polynomials change their sign n times in the interval $-1 \leq t = \cos\theta \leq 1$ or $0 \leq \theta \leq \pi$

Diagrams from Heiskanen & Moritz, p.24

THE GRAVITY FIELD OF THE EARTH

and the harmonic Y_5 will be positive (+ tve) and negative (- tve) in the following zones.

$\theta = 0°$	
	+ tve
24.5	
	- tve
58	
	+ tve
90	
	- tve
122	
	+ tve
155.5	
	- tve
$\theta = 180$	



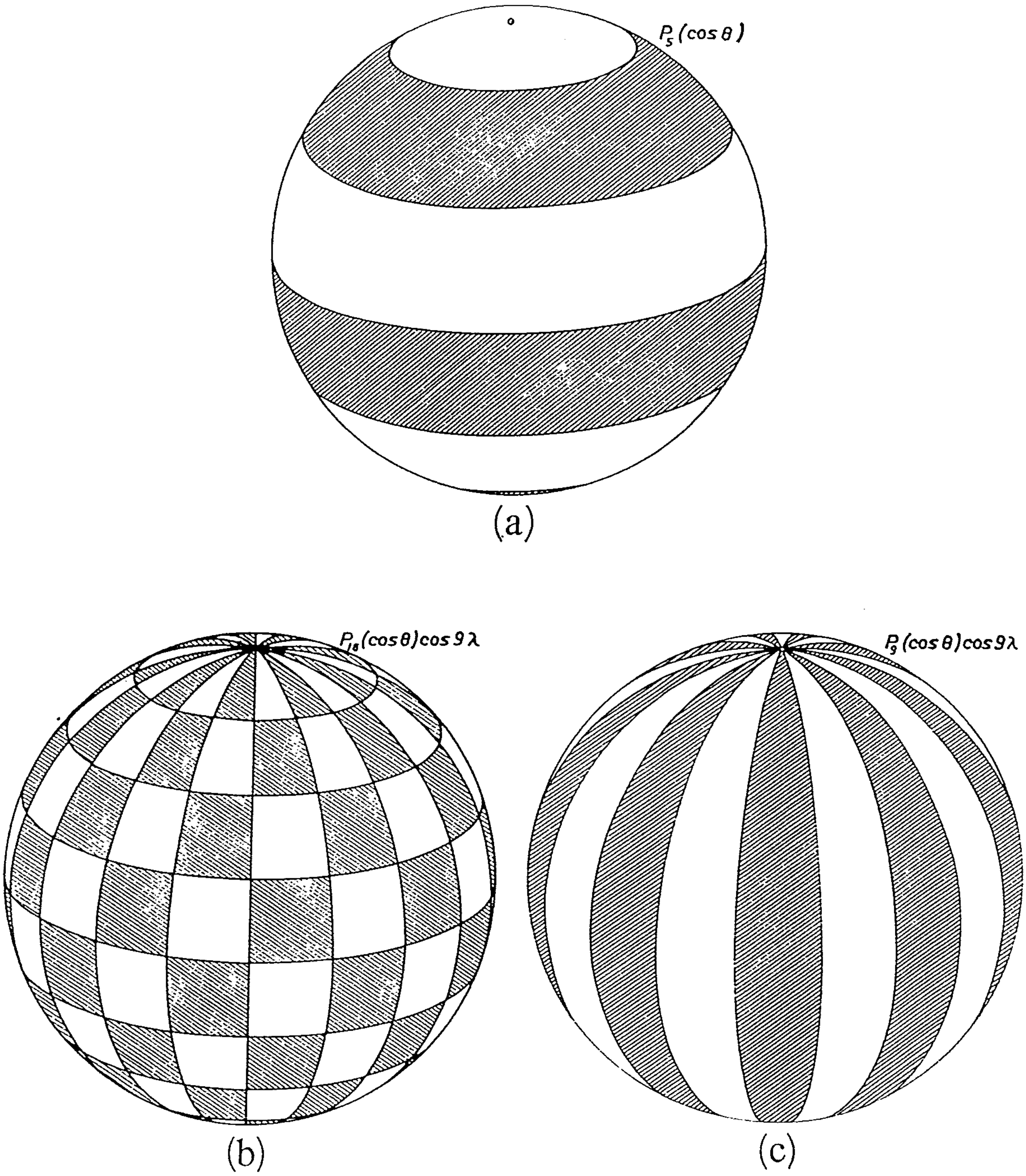
$0 \leq \theta \leq 180$

and hence the sphere is divided into six (6) zones symmetric about the equator ($\theta = 90°$). Notes that the zones are of varying "width" in a hemisphere.
See diagram (a) page 41

Associated Legendre functions where $m \neq 0$ will change their value (from positive to negative) $n - m$ times in the interval $0 \leq \theta \leq \pi$ and the trigonometric functions $\sin(m\lambda)$ and $\cos(m\lambda)$ have $2m$ zeroes in the interval $0 \leq \lambda \leq 2\pi$. Hence harmonics where $m \neq 0$ divide the sphere into alternating +tve and -tve compartments and are known as <u>TESSERAL HARMONICS</u> (see diag. (b), p 41)

Associated Legendre functions where $m = n$ are only functions of $\sin^m \theta$ which will always be positive for a value of $0 \leq \theta \leq \pi$ and hence harmonics where $m = n$ are functions of $\cos(m\lambda)$ and $\sin(m\lambda)$ only. These harmonics ($m = n$) divide the surface into alternating positive and negative sectors and are known as <u>SECTORIAL HARMONICS</u> (see diag. (c), p. 41).

THE GRAVITY FIELD OF THE EARTH.



The different kinds of spherical harmonics: (*a*) *zonal*, (*b*) *tesseral*, (*c*) *sectorial.*

Diagrams from Heiskanen & Moritz, p. 26

THE GRAVITY FIELD OF THE EARTH

"WORKING" FORMS OF THE GRAVITATIONAL POTENTIAL

Re-stating eqn (50) for the external gravitational potential (in conventional form)

$$\boxed{V(r,\theta,\lambda) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} P_n^m(t)\left\{ a_n^m \cos m\lambda + b_n^m \sin m\lambda \right\}} \quad \text{....(50)}$$

where $t = \cos\theta$

and (50) can be expressed as

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(t) \left\{ \frac{a_n^m}{r^{n+1}} \cos m\lambda + \frac{b_n^m}{r^{n+1}} \sin m\lambda \right\}$$

and the radius r can be "scaled" by dividing by a spherical earth radius $= a$ to give

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(t) \left\{ \frac{a_n^m}{\frac{r^{n+1}}{a^{n+1}} \cdot a^{n+1}} \cos m\lambda + \frac{b_n^m}{\frac{r^{n+1}}{a^{n+1}} \cdot a^{n+1}} \sin m\lambda \right\}$$

(noting that the "units" of V remain unchanged.)
which can be written as

$$\boxed{V = \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} P_n^m(t) \left\{ A_n^m \cos m\lambda + B_n^m \sin m\lambda \right\}} \quad \text{---- (69)}$$

where $A_n^m = \dfrac{a_n^m}{a^{n+1}}$, $B_n^m = \dfrac{b_n^m}{a^{n+1}}$

and $\dfrac{r^{n+1}}{a^{n+1}} = \left(\dfrac{r}{a}\right)^{n+1}$ and $\dfrac{1}{\left(\frac{r}{a}\right)^{n+1}} = \left(\dfrac{r}{a}\right)^{-(n+1)} = \left(\dfrac{a}{r}\right)^{n+1}$

a = radius of spherical Earth
a_n^m = coefficient

THE GRAVITY FIELD OF THE EARTH

Equation (69) can be written as

$$V(r,\theta,\lambda) = \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} D_n^m \qquad \text{.... (70)}$$

where $D_n^m = P_n^m(t)\left\{ A_n^m \cos m\lambda + B_n^m \sin m\lambda \right\}$

and the potential V can be expanded as

$$V = \frac{a}{r} D_0^0 + \left(\frac{a}{r}\right)^2 \left\{ D_1^0 + D_1^1 \right\} + \left(\frac{a}{r}\right)^3 \left\{ D_2^0 + D_2^1 + D_2^2 \right\} + \cdots$$

Taking a common factor of $^1/_r$ outside of the summation and re-arranging gives

$$V = \frac{1}{r}\left[aD_0^0 + \left(\frac{a}{r}\right) a D_0^0 \left\{ \frac{D_1^0}{D_0^0} + \frac{D_1^1}{D_0^0} \right\} + \left(\frac{a}{r}\right)^2 a D_0^0 \left\{ \frac{D_2^0}{D_0^0} + \frac{D_2^1}{D_0^0} + \frac{D_2^2}{D_0^0} \right\} + \cdots \right]$$

Now, the <u>mean value</u> of this <u>expanded spherical harmonic series</u> is the value for $n = 0$, the first term, and since the mean value of the Earth's gravitational potential is $\frac{GM}{r}$ then

$$\frac{GM}{r} = \frac{aD_0^0}{r} \quad \text{or} \quad aD_0^0 = GM$$

and
a = radius of spherical Earth
G = Univeral Gravitation constant
M = mass of Earth

and

$$V = \frac{1}{r}\left[GM + \frac{a}{r} GM \left\{ \frac{D_1^0}{D_0^0} + \frac{D_1^1}{D_0^0} \right\} + \left(\frac{a}{r}\right)^2 GM \left\{ \frac{D_2^0}{D_0^0} + \frac{D_2^1}{D_0^0} + \frac{D_2^2}{D_0^0} \right\} + \cdots \right]$$

and

$$V = \frac{GM}{r}\left[1 + \frac{a}{r}\left\{\frac{D_1^0}{D_0^0} + \frac{D_1^0}{D_0^0}\right\} + \left(\frac{a}{r}\right)^2\left\{\frac{D_2^0}{D_0^0} + \frac{D_2^1}{D_0^0} + \frac{D_2^0}{D_0^0}\right\} + \cdots\right]$$

and

$$\boxed{V = \frac{GM}{r}\left[1 + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^m(t)\left\{C_n^m \cos m\lambda + S_n^m \sin m\lambda\right\}\right]} \quad \text{---- (71)}$$

where

$$C_n^m = \frac{a}{GM} A_n^m = \frac{a_n^m}{GM(a)^n}$$

$$S_n^m = \frac{a}{GM} B_n^m = \frac{b_n^m}{GM(a)^n}$$

The gravitational potential is sometimes expressed as

$$\boxed{V = \frac{GM}{r}\left[1 - \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^m(t)\left\{J_n^m \cos m\lambda + K_n^m \sin m\lambda\right\}\right]} \quad \text{--- (72)}$$

where

$$J_n^m = -C_n^m$$

$$K_n^m = -S_n^m$$

THE GRAVITY FIELD OF THE EARTH

In summary, the Earth's external gravitational potential can be represented in the following ways

$$V(r,\theta,\lambda) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} P_n^m(t)\left\{ a_n^m \cos m\lambda + b_n^m \sin m\lambda \right\} \quad \cdots (50)$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} P_n^m(t)\left\{ A_n^m \cos m\lambda + B_n^m \sin m\lambda \right\} \quad \cdots (69)$$

$$= \frac{GM}{r}\left[1 + \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{n} \sum_{m=0}^{n} P_n^m(t)\left\{ C_n^m \cos m\lambda + S_n^m \sin m\lambda \right\} \right] \quad \cdots (71)$$

$$= \frac{GM}{r}\left[1 - \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{n} \sum_{m=0}^{n} P_n^m(t)\left\{ J_n^m \cos m\lambda + K_n^m \sin m\lambda \right\} \right] \quad \cdots (72)$$

where $\quad J_n^m = -C_n^m = -\dfrac{a}{GM} A_n^m = -\dfrac{a_n^m}{GM\,a^n}$

$$K_n^m = -S_n^m = -\frac{a}{GM} B_n^m = -\frac{b_n^m}{GM\,a^n}$$

$$P_n^m(t) = P_n^m(\cos\theta)$$

Note that each of these equations (expressed in conventional terms) can be expressed in <u>fully-normalized</u> terms $\bar{J}_n^m$, $\bar{K}_n^m$, $\bar{C}_n^m$, $\bar{S}_n^m$, $\bar{P}_n^m(t)$, etc by using the relationships given by equ's (67) and (68)

THE GRAVITY FIELD OF THE EARTH

THE PHYSICAL MEANING OF SOME LOWER DEGREE HARMONIC TERMS

In harmonic series the first term ($n=0$) is the mean value of the function and subsequent terms are deviations from the mean value. In the case of a SPHERICAL HARMONIC EXPANSION of the Earth's external gravitational potential the first term ($n=0$) equals GM/r which is the gravitational potential of a sphere whose mass is regarded as homogeneous and of uniform density distribution. The subsequent terms in the expansion (deviations from the mean value) represent harmonic deviations from the mean. The magnitude of the coefficients (say C_n^m and S_n^m in eq. 71) are related to uneven mass distributions or density distributions, thus where the potential is positive with respect to the mean value, there is a mass excess. Similarly, there is a mass deficiency where the deviation from the mean is negative.

Equation (71), expressed in conventional harmonics is

$$V = \frac{GM}{r}\left[1 + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_n^m(t)\left\{C_n^m \cos m\lambda + S_n^m \sin m\lambda\right\}\right] \quad \ldots\ (71)$$

where n and m are zero or positive integers
C_n^m, S_n^m are potential coefficients
$P_n^m(t)$ are associated Legendre functions
a radius of spherical earth
GM product of Universal Gravitational constant G and mass of Earth M
$t = \cos\theta$

THE GRAVITY FIELD OF THE EARTH

Expanding (71) for degrees $n = 1, 2$ gives

DEGREE 1 $(n = 1)$

$m = 0$ $\quad P_1^0(t)\left\{ C_1^0 \cos(0) + S_1^0 \sin(0) \right\} = P_1^0(t)\left\{C_1^0\right\}$

$m = 1$ $\quad P_1^1(t)\left\{ C_1^1 \cos\lambda + S_1^1 \sin\lambda \right\}$

DEGREE 2 $(n = 2)$

$m = 0$ $\quad P_2^0(t)\left\{ C_2^0 \right\}$

$m = 1$ $\quad P_2^1(t)\left\{ C_2^1 \cos\lambda + S_2^1 \sin\lambda \right\}$

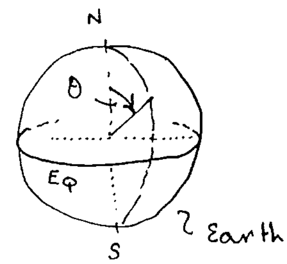
$m = 2$ $\quad P_2^2(t)\left\{ C_2^2 \cos 2\lambda + S_2^2 \sin 2\lambda \right\}$

Now consider the individual harmonic terms.

DEGREE $n=1$, ORDER $m = 0$ $\quad P_1^0(t)\left\{ C_1^0 \right\}$

now, since $P_1^0(t) = t$ (see p. 25) and $t = \cos\theta$ this harmonic term is

$$n=1,\ m=0 \qquad P_1^0(t)\left\{C_1^0\right\} = C_1^0 \cos\theta$$



Since $\cos\theta$ is POSITIVE in the northern hemisphere and NEGATIVE in the southern hemisphere, then the presence of the term C_1^0 would indicate a mass excess in the northern hemisphere and a mass deficiency in the southern hemisphere

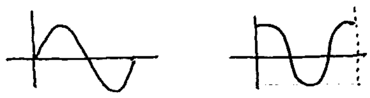
THE GRAVITY FIELD OF THE EARTH

This would mean that the centre of mass would NOT coincide with the coordinate origin. Since this is a basic requirement (COORD ORIGIN = CENTRE OF MASS) this harmonic term is NOT ADMISSIBLE in the expression for potential. To achieve this, the coefficient C_1^0 is set to zero.

DEGREE $n = 1$ ORDER $m = 1$ $\quad P_1^1(t)\left\{C_1^1 \cos\lambda + S_1^1 \sin\lambda\right\}$

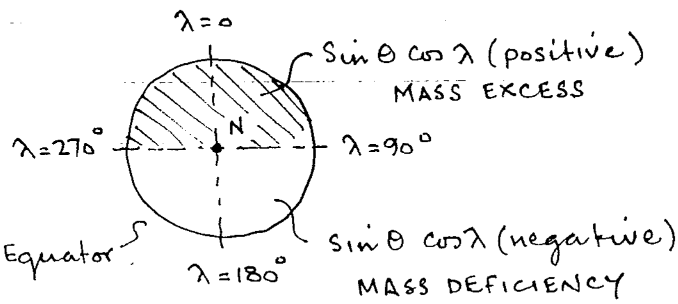
Now, since $P_1^1(t) = (1-t^2)^{1/2} = \sin\theta$ the harmonic terms are

$$C_1^1 \sin\theta \cos\lambda \quad \text{and} \quad S_1^1 \sin\theta \sin\lambda$$



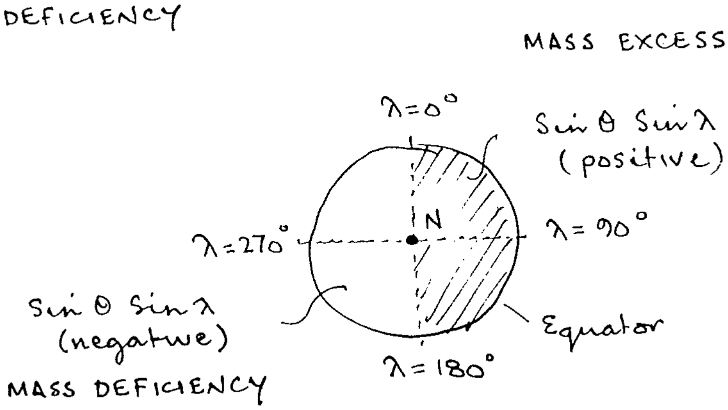
Sin θ $\qquad$ Cos θ

Now the sin θ term will be positive in the northern and southern hemispheres. Cos λ will be negative between λ = 90° and λ = 270° and positive between λ = 270° and λ = 90°



harmonic term
$\leftarrow$ $C_1^1 \sin\theta \cos\lambda$

harmonic term
$S_1^1 \sin\theta \sin\lambda$ $\rightarrow$



THE GRAVITY FIELD OF THE EARTH

Similarly, $\sin\lambda$ is positive between $\lambda = 0°$ and $\lambda = 180°$ and negative between $\lambda = 180°$ and $\lambda = 0°$

Using the same reasoning as before, if these terms are allowed they would mean the CENTRE OF MASS would NOT coincide with the COORDINATE ORIGIN. Hence these harmonic terms are not admissible in the expression for potential. To achieve this, the coefficients C_1^1 and S_1^1 are set to zero.

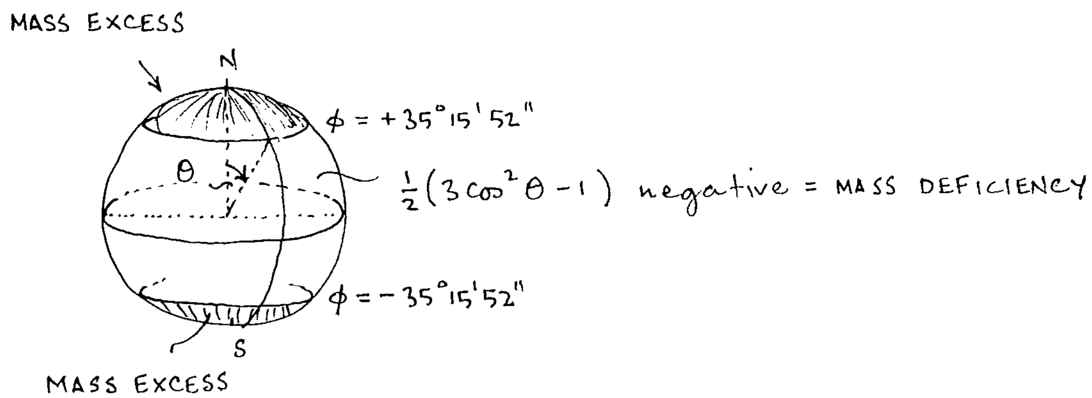
DEGREE $n = 2$ ORDER $m = 0$ $\quad P_2^0(t)\ \{C_2^0\}$

Now, since $P_2^0(t) = \frac{1}{2}(3t^2 - 1) = \frac{1}{2}(3\cos^2\theta - 1)$
and the harmonic term is

$$C_2^0\ \tfrac{1}{2}(3\cos^2\theta - 1)$$

which is a ZONAL HARMONIC since it is independent of the longitude λ.

The term $\frac{1}{2}(3\cos^2\theta - 1)$ is negative for $54°44'08'' < \theta < 125°15'52''$ and positive for $\theta < 54°44'08''$ and $\theta > 125°15'52''$



THE GRAVITY FIELD OF THE EARTH

Since it is known that the Earth is slightly squashed at the poles, the coefficient C_2^0 is negative to reflect the actual mass deficiency in the polar regions.

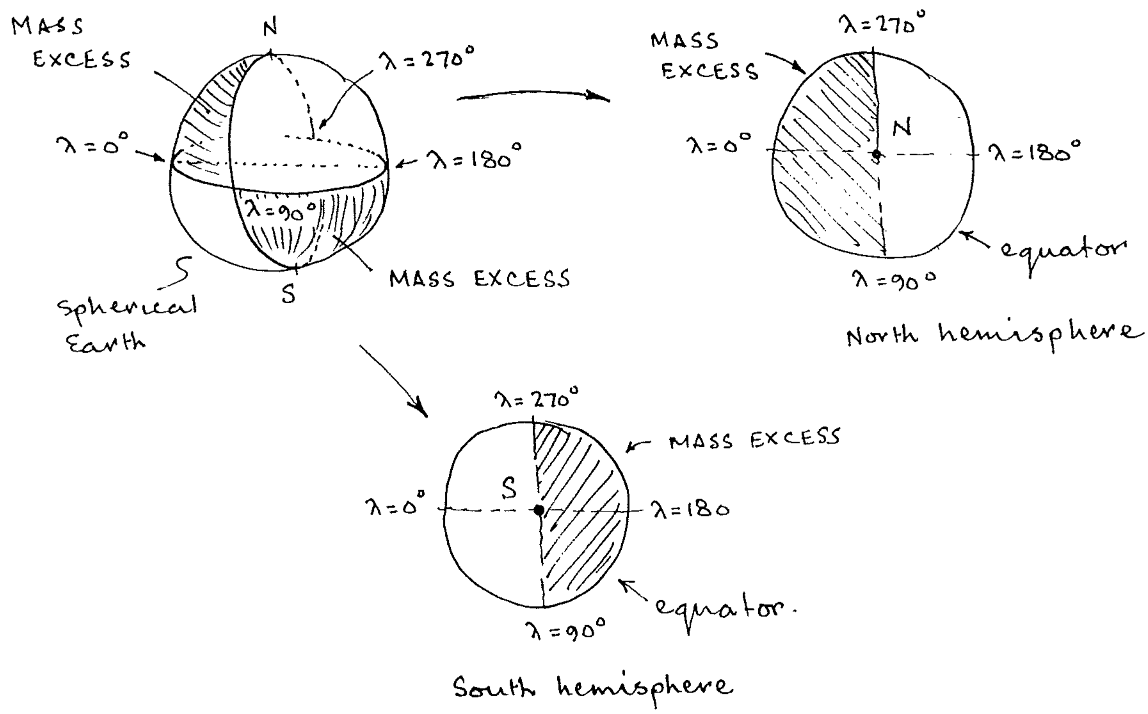
[The magnitude of the C_2^0 coefficient is approx. 100 times larger than any other harmonic coefficient.]

DEGREE $n = 2$ ORDER $m = 1$ $\quad P_2^1(t)\left\{ C_2^1 \cos\lambda + S_2^1 \sin\lambda \right\}$

Now since $P_2^1(t) = 3t\,(1-t^2)^{1/2} = 3\cos\theta\,\sin\theta$ the harmonic terms are

$$C_2^1\, 3\cos\theta\,\sin\theta\,\cos\lambda \quad \text{and} \quad S_2^1\, 3\cos\theta\,\sin\theta\,\sin\lambda$$

The term $3\cos\theta\,\sin\theta\,\cos\lambda$ is positive when $\cos\theta$ is positive ($0° < \theta \leq 90°$) and $\cos\lambda$ is positive ($0° < \lambda < 90°$ and $270° < \lambda < 0°$) (noting that $\sin\theta$ is always positive) and also when $\cos\theta$ is negative ($90° < \theta \leq 180°$) and $\cos\lambda$ is negative ($90° < \lambda \leq 270°$)



This harmonic term represents opposite northern and southern quadrants of mass excesses and deficiencies. This would mean that the axis of inertia would revolve around the axis of revolution of the Earth causing the Earth to wobble. This wobble (commonly called "precession") does actually occur, hence the C_2^1 coefficient cannot equal zero.

Using similar reasoning, the harmonic term $S_2^1\, 3\cos\theta \sin\theta \sin\lambda$ can also be analyzed. The geometry is similar except the quadrants of mass excess and deficiency are advanced by $90°$ in longitude.

The term S_2^1 cannot equal zero.

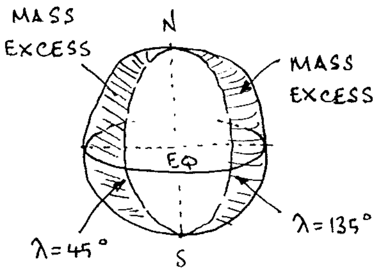
DEGREE $n=2$ ORDER $m=2$ $\quad P_2^2(t)\left\{ C_2^2 \cos 2\lambda + S_2^2 \sin 2\lambda \right\}$

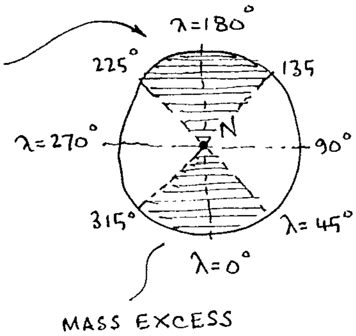
Now, since $P_2^2(t) = 3(1-t^2) = 3\sin^2\theta$ the harmonic terms are

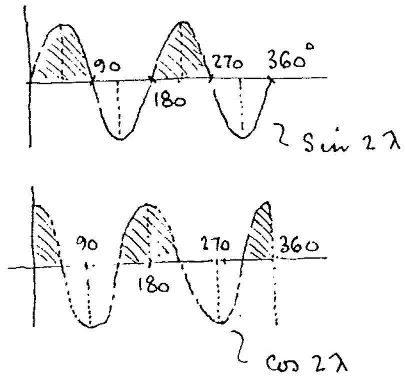
$$C_2^2\, 3\sin^2\theta \cos 2\lambda \quad \text{and} \quad S_2^2\, 3\sin^2\theta \sin 2\lambda$$

These are SECTORIAL HARMONICS.

The term $3\sin^2\theta\cos 2\lambda$ is positive for all values of θ $(0° \le \theta \le 180°)$ and for $315° < \lambda < 45°$ and $135° < \lambda < 225°$



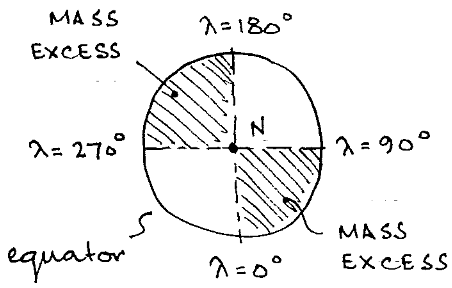




THE GRAVITY FIELD OF THE EARTH

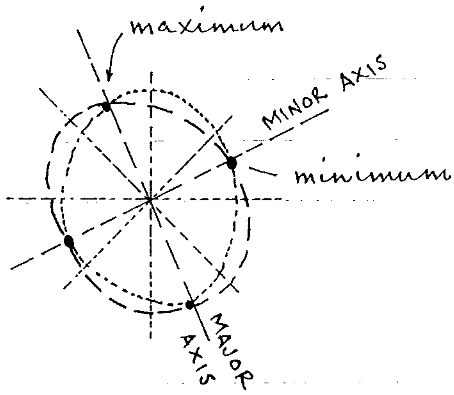
The term $3\sin^2\theta\cos 2\lambda$ indicates ellipticity of the equator with mass excesses in the sectors $315^\circ < \lambda < 45^\circ$ and $135^\circ < \lambda < 225^\circ$.

Using similar reasoning the term $3\sin^2\theta\sin 2\lambda$ indicates ellipticity of the equator due to mass excesses in the sectors $0^\circ < \lambda < 90^\circ$ and $180^\circ < \lambda < 270^\circ$



Both harmonic terms represent ellipticity of the Earth's equator in different directions.

The combination of the two terms gives the directions of the semi-axes of the Earth's equatorial ellipse.



The direction of the major-axis is at the point where their sum is a maximum and the direction of the minor-axis is advanced 90° from the major axis.

THE GRAVITY FIELD OF THE EARTH

In summary, the coefficients

$n=1 \quad m=0 \qquad C_1^0 = 0$ set to zero to enforce the coordinate origin to coincide with the Earth's centre of mass.

$S_1^0 = 0$ since $\sin(m\lambda) = 0$

$n=1 \quad m=1 \qquad \left.\begin{array}{l} C_1^1 = 0 \\ S_1^1 = 0 \end{array}\right\}$ Both set to zero to enforce coord origin coincident with Earth's centre of mass.

give rise to the following form of the SPHERICAL HARMONIC EXPANSION OF THE EARTH'S GRAVITATIONAL POTENTIAL

$$\boxed{V = \frac{GM}{r}\left[1 + \sum_{n=2}^{N}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} \bar{P}_n^m(t)\left\{\bar{C}_n^m \cos m\lambda + \bar{S}_n^m \sin m\lambda\right\}\right]} \quad \text{---}(73)$$

where $\bar{P}_n^m(t)$, $\bar{C}_n^m$ and $\bar{S}_n^m$ are fully normalized and the summation for n begins at $n=2$ and ends at N — the maximum degree of the GEOPOTENTIAL MODEL.

The current geopotential model, EGM96 (Earth Gravitational Model 1996) has coefficients $\bar{C}_n^m$ and $\bar{S}_n^m$ to degree and order 360 (approx 130,000 terms).

THE GRAVITY FIELD OF THE EARTH

COMPUTATION OF THE GRAVITATIONAL POTENTIAL V

$$V = \frac{GM}{r}\left[1 + \sum_{n=2}^{N}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} \bar{P}_n^m(t)\left\{\bar{C}_n^m \cos m\lambda + \bar{S}_n^m \sin m\lambda\right\}\right] \qquad \cdots(73)$$

Consider Melbourne: $\phi = -37°48'00''$, $\lambda = 144°58'00''$ E, $h = 0.000$ on the GRS 80 ellipsoid ($a = 6378137.000$ m, $f = {}^1/_{298.257\,222\,100\,881}$) with Cartesian coords

$$\begin{aligned}
X &= -4131\,810.563 \text{ m.} \\
Y &= 2896\,708.708 \text{ m.} \\
Z &= -3887\,927.165 \\
r &= \sqrt{X^2+Y^2+Z^2} &&= 6370\,145.800 \text{ m.} \\
\psi &= \sin^{-1}\frac{Z}{R} &&= -37°36'49.5994'' \quad \text{geocentric latitude} \\
\theta &= 90 - \psi &&= 127°36'49.5994'' \\
t &= \cos\theta &&= -0.610\,335\,664 \\
u &= \sin\theta &&= 0.792\,142\,902
\end{aligned}$$

To compute the potential V, the FULLY-NORMALIZED Legendre functions $\bar{P}_n^m(t)$ must be computed.

From the relationships between CONVENTIONAL and FULLY-NORMALIZED Legendre functions (see. eq 67)

$$\bar{P}_n^m(t) = \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}}\, P_n^m(t) \qquad \text{for } m > 0$$

$$\bar{P}_n(t) = \sqrt{2n+1}\, P_n(t) \qquad \cdots(74)$$

$$\bar{P}_m^m(t) = \sqrt{\frac{2(2m+1)}{(2m)!}}\, P_m^m(t)$$

Equations (74) can be used to compute FULLY-NORMALIZED Legendre functions from the CONVENTIONAL Legendre functions already calculated and tabulated on p.30

Alternatively, RECURRENCE formulae for FULLY-NORMALIZED Legendre functions may be deduced by using eq.(67) and equations (56a) and (56b) to give

$$\bar{P}_n^m(t) = \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\; t\, \bar{P}_{n-1}^m(t) - \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}}\; \bar{P}_{n-2}^m(t)$$

$$\bar{P}_m^m(t) = \sqrt{\frac{2m+1}{2m}}\; u\, \bar{P}_{m-1}^{m-1}(t)$$

$$\bar{P}_{n+1}^n(t) = \sqrt{2n+3}\; t\, \bar{P}_n^n(t) \qquad \text{.... (75)}$$

Equations (75) require the "seeds" $\bar{P}_0^0(t) = 1$, $\bar{P}_1^0(t) = \sqrt{3}\,t$, $\bar{P}_1^1(t) = \sqrt{3}\,u$

n \ m	0	1	2	3	4
0	1.000 000 000				
1	-1.057 132 380	1.372 031 753			
2	0.131 401 270	-1.872 483 126	1.215 129 891		
3	0.918 376 596	1.107 009 935	-1.962 187 352	1.039 679 867	
4	-1.244 463 381	0.449 985 655	1.691 696 751	-1.903 661 105	0.873 533 230

Fully-normalized Associated Legendre functions to degree (n) and order (m) equal to 4 for $t = \cos\theta = -0.610\,335\,664$ ($\theta = 127°\,36'\,49.5994''$)

THE GRAVITY FIELD OF THE EARTH

Fully-normalized coefficients for the Geopotential Models: EGM96 and OSU91A1F

c EGM96 geopotential model to degree(n) and order(m) = 4.
c The coefficients $\bar{C}(n,m)$ and $\bar{S}(n,m)$ are fully-normalized.
c The geopotential model has GM = 3986004.415E+08 m**3/s**2
c a = 6378136.3 m
c
c

c n	m	$\bar{C}(n,m)$	$\bar{S}(n,m)$
0	0	0.000000000000E+00	0.000000000000E+00
1	0	0.000000000000E+00	0.000000000000E+00
1	1	0.000000000000E+00	0.000000000000E+00
2	0	-0.484165371736E-03	0.000000000000E+00
2	1	-0.186987635955E-09	0.119528012031E-08
2	2	0.243914352398E-05	-0.140016683654E-05
3	0	0.957254173792E-06	0.000000000000E+00
3	1	0.202998882184E-05	0.248513158716E-06
3	2	0.904627768605E-06	-0.619025944205E-06
3	3	0.721072657057E-06	0.141435626958E-05
4	0	0.539873863789E-06	0.000000000000E+00
4	1	-0.536321616971E-06	-0.473440265853E-06
4	2	0.350694105785E-06	0.662671572540E-06
4	3	0.990771803829E-06	-0.200928369177E-06
4	4	-0.188560802735E-06	0.308853169333E-06

c
c
c
c OSU91A1F geopotential model to degree(n) and order(m) = 4.
c The coefficients $\bar{C}(n,m)$ and $\bar{S}(n,m)$ are fully-normalized.
c The geopotential model has GM = 3986004.360E+08 m**3/s**2
c a = 6378137.0 m
c
c

c n	m	$\bar{C}(n,m)$	$\bar{S}(n,m)$
0	0	0.000000000000E+00	0.000000000000E+00
1	0	0.000000000000E+00	0.000000000000E+00
1	1	0.000000000000E+00	0.000000000000E+00
2	0	-0.484165532804E-03	0.000000000000E+00
2	1	0.857179552165E-12	0.289607376372E-11
2	2	0.243815798120E-05	-0.139990174643E-05
3	0	0.957139401177E-06	0.000000000000E+00
3	1	0.202968777310E-05	0.249431310090E-06
3	2	0.904648670700E-06	-0.620437816800E-06
3	3	0.720295507400E-06	0.141470959443E-05
4	0	0.540441629840E-06	0.000000000000E+00
4	1	-0.535373285210E-06	-0.474065010407E-06
4	2	0.350729847400E-06	0.663967363224E-06
4	3	0.991080200230E-06	-0.202148896490E-06
4	4	-0.190576531700E-06	0.309704028950E-06

c
c end of data

THE GRAVITY FIELD OF THE EARTH

Expanding (73) gives

$$V = \frac{GM}{r}\Bigg[1 + \left(\frac{a}{r}\right)^2 \Big\{ \bar{P}_2^0 (\bar{C}_2^0) + \bar{P}_2^1 (\bar{C}_2^1 \cos\lambda + \bar{S}_2^1 \sin\lambda) + \bar{P}_2^2 (\bar{C}_2^2 \cos 2\lambda + \bar{S}_2^2 \sin 2\lambda) \Big\}$$

$$+ \left(\frac{a}{r}\right)^3 \Big\{ \bar{P}_3^0 (\bar{C}_3^0) + \bar{P}_3^1 (\bar{C}_3^1 \cos\lambda + \bar{S}_3^1 \sin\lambda) + \bar{P}_3^2 (\bar{C}_3^2 \cos 2\lambda + \bar{S}_3^2 \sin 2\lambda) + \bar{P}_3^3 (\bar{C}_3^3 \cos 3\lambda + \bar{S}_3^3 \sin 3\lambda) \Big\}$$

$$+ \left(\frac{a}{r}\right)^4 \Big\{ \bar{P}_4^0 (\bar{C}_4^0) + \bar{P}_4^1 (\bar{C}_4^1 \cos\lambda + \bar{S}_4^1 \sin\lambda) + \bar{P}_4^2 (\bar{C}_4^2 \cos 2\lambda + \bar{S}_4^2 \sin 2\lambda) + \bar{P}_4^3 (\bar{C}_4^3 \cos 3\lambda + \bar{S}_4^3 \sin 3\lambda) + \bar{P}_4^4 (\bar{C}_4^4 \cos 4\lambda + \bar{S}_4^4 \sin 4\lambda) \Big\}\Bigg]$$

and for the EGM96 Geopotential Model (coefficients $\bar{C}_n^m$ and $\bar{S}_n^m$ given on p. 56)

$$GM = 3986\,004.415 \times 10^8 \ \text{m}^3/\text{s}^2$$

$$a = 6378\,136.3 \ \text{m},$$

and for Melbourne

$$r = 6370\,145.800 \ \text{m}.$$

$$\lambda = 144°\,58'\,00''$$

$$V = \frac{GM}{r}\Bigg[1 + \left(\frac{a}{r}\right)^2 (-6.101\,160 \times 10^{-5}) + \left(\frac{a}{r}\right)^3 (-9.350\,518 \times 10^{-7}) + \left(\frac{a}{r}\right)^4 (-1.616\,713 \times 10^{-6})\Bigg]$$

$$= \frac{GM}{r} (0.999\,936\,272)$$

$$\underline{V = 62\,569\,217.71} \quad \text{m}^2/\text{s}^2$$

THE GRAVITY FIELD OF THE EARTH

COMPUTATION OF GRAVITY POTENTIAL W

Referring to eq.(27), the GRAVITY POTENTIAL W is the sum of the GRAVITATIONAL POTENTIAL V and the CENTRIFUGAL POTENTIAL (or ROTATIONAL POTENTIAL) Q

$$\underbrace{W}_{\text{gravity potential}} = \underbrace{V}_{\text{Gravitational potential}} + \underbrace{Q}_{\text{Rotational Potential}} \quad \text{--- (27)}$$

Now from (24)

$$Q = \frac{\omega^2}{2}\left(X^2 + Y^2\right) \quad \text{---- (24)}$$

where ω (omega) is the angular velocity of the Earth.
and

$$\omega = 7292115 \times 10^{-11} \text{ radians/sec.}$$

For Melbourne $\phi = S.\,37°48'$, $\lambda = 144°58'$, $h = 0.000$ m on the GRS 80 ellipsoid

$$X = -4131\,810.563 \text{ m.}$$

$$Y = 2896\,708.708 \text{ m.}$$

$$Z = -3887\,927.165 \text{ m.}$$

Eq (27) can also be expressed as

$$Q = \frac{\omega^2}{2}\left(r \cos\psi\right)^2 \quad \text{--- (75)}$$

where ψ is the GEOCENTRIC LATITUDE

$$\underline{Q = 67\,699.09 \text{ m}^2/\text{s}^2} \quad \text{and} \quad \underline{\underline{W = 62\,636\,916.80}} \text{ m}^2/\text{s}^2$$

THE GRAVITY FIELD OF THE EARTH

REFERENCES

Heiskanen & Moritz, 1967. "Physical Geodesy". W.H. Freeman & Co. London. (Chapters 1 and 2.)

Bomford, G. (1980). "Geodesy". 4th Ed. Clarendon Press, Oxford. (Chapter 6 and Appendix H)

Defense Mapping Agency (1983). "Geodesy for the Layman". DMA Technical Report DMA TR 80-003, Defense Mapping Agency Aerospace Centre, St. Louis AFS, Mo 63118.